STATISTICS 383
HOMEWORK ASSIGNMENT 8
DUE WEDNESDAY, JUNE 5 2019

Problem 1. First-passage time process. Let $\{W_t\}_{t\geq 0}$ be a standard one-dimensional Wiener process, and for each $a \geq 0$ define τ_a to be the first time t that $W_t = a$.

(A) Use the strong Markov property for Brownian motion to show that the stochastic process $\{\tau_a\}_{a\geq 0}$ has stationary, independent increments (i.e., it is a *Lévy process*).

(B) Check that the sample paths $a \mapsto \tau_a$ are nondecreasing and *left*-continuous. (Note: A Lévy process with nondecreasing paths is called a *subordinator*.)

(C) Show that for every $a > 0$, the process $\{a^{-2}\tau_{as}\}_{s\geq 0}$ has the same law as the process $\{\tau_s\}_{s\geq 0}$. HINT: Brownian scaling.

(D) Prove that with probability one, the set $\{a \geq 0 : \tau_{a+} - \tau_a > 0\}$ of jump discontinuities is countable and dense. HINT: The hard part is showing that with probability 1 the discontinuities are dense. For this the result of part (C) might be useful.

Problem 2. Two-dimensional Brownian motion: first-passage distribution. Let $Z_t = (X_t, Y_t)$ be a two-dimensional Brownian motion started at the origin $(0,0)$ (that is, the coordinate processes X_t and Y_t are independent standard one-dimensional Wiener processes).

(A)Prove that for each real θ, the process $\exp\{\theta X_t + i\theta Y_t\}$ is a martingale relative to any admissible filtration.

(B) Deduce the corresponding Wald identity for the first passage time $\tau(a) = \min\{t : W_t = a\}$, for $a > 0$.

(C) What does this tell you about the distribution of $Y_{\tau(a)}$?

Problem 3. Let $\{W_t\}_{t\geq 0}$ be a standard one-dimensional Wiener process, and for each $t \geq 0$ let $M_t = \max_{s\leq t} W_s$.

(A) Use the reflection principle to find the joint distribution of (W_t, M_t). (The answer is given in Corollary 5 of the notes; your job is to supply the derivation.)

(B) Use the result of part (A) to conclude that for every t, the distribution of $M_t - W_t$ is the same as that of $|W_t|$.

Problem 4. For each $k = 0, 1, 2, \ldots$ define stopping times $T_{k,1}, T_{k,2}, \ldots$ as follows:

$$T_{k,0} = 0 \quad \text{and} \quad T_{k,m+1} = \min\{t > T_{k,m} : |W_t - W_{T_{k,m}}| = 2^{-k}\}.$$

are the jump times of the kth level embedded simple random walk. For each $s > 0$ define $N_k(s)$ to be $\max\{m : T_{k,m} < s\}$; thus, $N_k(s)$ is the number of jumps made by the kth level embedded random walk by time s. Show that for each s, the sequence of random variables $N_k(s)/4^s$ converges *in probability* to s, that is, for every $\varepsilon > 0$

$$\lim_{k\to\infty} P\{|N_k(s)/4^k - s| > \varepsilon\} = 0.$$

In fact, the convergence holds *almost surely*; if you can prove this, all the better.

HINTS: (i) For each k the random variables $(T_{k,m+1} - T_{k,m})_{m=0,1,2,\cdots}$ are independent, identically distributed. (ii) For each k the random variable $4^k T_{k,1}$ has the same distribution as $T_{1,1}$.

Bonus Problem.

Problem 5. Let $\{W_t\}_{t\geq 0}$ be a standard one-dimensional Wiener process, and for each pair $0 \leq s \leq t$ define $M(s,t)$ to be the maximum value attained by W_r for $s \leq r \leq t$.

(A) Show that with probability one, for every pair of *rational* $0 \leq s < t$,

$$M(s,t) > \max(W_s, W_t).$$

(B) Conclude that with probability one, the local maxima of the Brownian path $t \mapsto W_t$ are dense in $[0,\infty)$. Also, prove that the set of *times* t at which the Brownian path has local maxima are dense in $[0,\infty)$. NOTE: By definition, a local maximum occurs at any time t such that for some $\varepsilon > 0$,

$$W_t \geq \max_{s\in[t-\varepsilon,t+\varepsilon]} W_s.$$

(C) Prove that with probability one, for every rational pair $0 \leq s < t$ the maximum value $M(s,t)$ of the Brownian path on the time interval $[s,t]$ is attained at a unique time $r \in (s,t)$. Thus, with probability one, the local maxima of the Brownian path are distinct.