STATISTICS 383: MEASURE-THEORETIC PROBABILITY II
HOMEWORK ASSIGNMENT 4
DUE MONDAY APRIL 30, 2018

Problem 1. *Martingales with bounded increments.* Let $\{S_n\}_{n\geq 0}$ be a martingale with respect to the filtration $\{\mathcal{F}_n\}_{n\geq 0}$ such that $S_0 = 0$ and such that the *martingale differences* $\xi_n := S_n - S_{n-1}$ are uniformly bounded. Assume for convenience that $|\xi_n| \leq 1$.

(A) For any integer $m \geq 1$ define

$$\begin{aligned} \tau_m &= \min \{n \geq 0 \, : \, S_n \leq -m\}, \\ &= +\infty \quad \text{if } S_{-n} > -m \; \forall \, n \in \mathbb{N}. \end{aligned}$$

Prove that $\lim_{n\to\infty} S_{n\wedge\tau_m} := Z_m$ exists and is finite almost surely. HINT: Recall that every *nonnegative* martingale is L^1-bounded, and therefore converges almost surely.

(B) Conclude from (A) that on the event $\left\{\sup_{n\geq 1} S_n < \infty\right\}$, the limit $\lim_{n\to\infty} S_n$ exists and is finite almost surely. (By replacing the martingale S_n by the martingale $-S_n$, you should then deduce that on the event $\{\inf_{n\geq 1} S_n > -\infty\}$, the limit $\lim_{n\to\infty} S_n$ exists and is finite almost surely.)

(C) Now conclude that with probability one, *either* $\lim_{n\to\infty} S_n$ exists and is finite *or*

$$\begin{aligned} \limsup_{n\to\infty} S_n &= \infty \quad \text{and} \\ \liminf_{n\to\infty} S_n &= -\infty. \end{aligned}$$

(D) Show that the conclusion of (C) is not in general true without the hypothesis that the increments ξ_n are bounded by exhibiting a martingale S_n such that $\lim_{n\to\infty} S_n = +\infty$ almost surely. HINT: Try finding an example with *independent* increments ξ_n such that each ξ_n takes one of only two possible values.

(E) Let $\{A_n\}_{n\geq 1}$ be a sequence of events such that $A_n \in \mathcal{F}_n$ for every $n \geq 1$. Prove that the events

$$\left\{\sum_{n=1}^{\infty} \mathbf{1}_{A_n} = \infty\right\} \quad \text{and} \quad \left\{\sum_{n=1}^{\infty} P(A_n \,|\, \mathcal{F}_{n-1}) = \infty\right\}$$

differ by at most an event of probability 0. HINT: Deduce this from (C). You will need to find an appropriate martingale with bounded increments.

NOTE: This clearly generalizes the usual Borel-Cantelli Lemma. It is quite a useful generalization because it does not require the events A_n to be independent.

Problem 2. Recall that in the *Polya urn scheme* one repreatedly samples at random from an urn containing a finite number of balls that are identical except for their colors. Whenever a ball is drawn, it is then returned to the urn along with a new ball of the same color.

(A) Suppose that at time $n = 0$ the urn has $R_0 = 1$ red ball and $B_0 = 1$ blue ball. Let R_n and B_n be the numbers of red and blue balls in the urn after n draws (thus, $R_n + B_n = 2 + n$). Show that R_n has the uniform distribution on the set

$$1, 2, 3, \cdots, n + 1$$

Conclude that $\Theta_\infty := \lim_{n\to\infty} R_n/(n + 2)$ is uniformly distributed on the unit interval $[0, 1]$.

In parts (B)-(C) suppose that at time $n = 0$ the urn has 1 red, 1 blue, and 1 green ball. Let R_n, B_n, G_n be the numbers of red, blue, and green balls in the urn after n draws (so $R_n + B_n + G_n = n + 3$.

(B) Show that for all integer pairs j, k satisfying $1 \leq j < k \leq n + 2$,

$$P\{R_n = j \text{ and } R_n + B_n = k\} = 1/\binom{n+2}{2}.$$

Observe that this is equivalent to randomly selecting 2 numbers *without replacement* from the list

$$1, 2, 3, \cdots, n + 2.$$

(C) Show that $\Theta^R_\infty := \lim R_n/(n+3)$ and $\Theta^B_\infty := \lim B_n/(n+3)$ exist with probability one. Conclude from part (B) that the joint distribution of Θ^R_∞ and $\Theta^R_\infty + \Theta^B_\infty$ is the same as that of $U_{(1)}$ and $U_{(2)}$, the order statistics of a sample of size two from the uniform distribution on $[0, 1]$.

(D) Now suppose that at time 0 the urn has 2 red and 1 blue ball. What is the distribution of $\Theta_\infty := \lim R_n/(n + 3)$?

Bonus Problem.

Problem 3. "Survivor". This is a game in which members of a finite population S_0 are successively removed until only one remains. Not all individuals in the population are equally vulnerable, however: instead, it is assumed that each individual x is assigned a *fitness* $f(x) > 0$ that partially determines his/her chance of removal at each stage of the game. The game is played as follows.

Let S_n be the set of individuals who have survived the first n rounds of play. Given S_n, let A_n be a random subset of S_n (chosen uniformly among all subsets of S_n), and let

$B_n = S_n \setminus A_n$ be the complementary subset. Let U_n be a random variable uniformly distributed on the unit interval (and independent of all prior random choices), and set

$$
\begin{aligned}
S_{n+1} &= A_n && \text{if } U_n \leq \frac{\sum_{x \in A_n} f(x)}{\sum_{x \in S_n} f(x)}; \\
S_{n+1} &= B_n && \text{if } U_n > \frac{\sum_{x \in A_n} f(x)}{\sum_{x \in S_n} f(x)}.
\end{aligned}
$$

(A) Show that with probability 1, eventually S_n will be a singleton $\{X\}$.

(B) Fix $x \in S_0$, and for each $n \geq 0$ define

$$
\begin{aligned}
Z_n^x &= \frac{f(x)}{\sum_{y \in S_n} f(y)} && \text{if } x \in S_n, \\
Z_n^x &= 0 && \text{otherwise.}
\end{aligned}
$$

Show that the sequence $\{Z_n^x\}_{n \geq 0}$ is a martingale.

(C) What is the distribution of X?