STATISTICS 383: MEASURE-THEORETIC PROBABILITY II
HOMEWORK ASSIGNMENT 3
DUE MONDAY APRIL 22, 2019

In these problem, assume that $(\Omega, \mathcal{F}, P)$ is a fixed probability space, and that $\{\mathcal{F}_n\}_{n\geq 0}$ is a filtration of this probability space. Any martingale, sub-martingale, stopping time, etc., should be understood to be a martingale, sub-martingale, stopping time, etc., with respect to the filtration $\{\mathcal{F}_n\}_{n\geq 0}$.

Problem 1. Show that every submartingale $\{X_n\}_{n\geq 0}$ can be witten as

$$X_n = Y_n + Z_n$$

where $\{Y_n\}_{n\geq 0}$ is a martingale and $\{Z_n\}_{n\geq 1}$ is a *predictable* sequence satisfying

$$0 = Z_0 \leq Z_1 \leq Z_2 \leq \cdots .$$

NOTES: (a) There is a similar decomposition for *super*-martingales. What is it? (b) This problem shows that the Martingale Convergence Theorem implies that every L^1-bounded *sub*-martingale converges almost surely.

Problem 2. Give an example of a martingale $\{X_n\}_{n\geq 0}$ such that $\lim_{n\to\infty} X_n = -\infty$ almost surely.

Problem 3. Let $\{X_n\}_{n\geq 0}$ be a martingale and set

$$M_n = \max_{k\leq n} |X_k| \quad \text{and}$$
$$M = \sup_{n\geq 1} M_n$$

(A) Show that $P\{M_n \geq t\} \leq t^{-1} E|X_n| \mathbf{1}_{\{M_n \geq t\}}$.

(B) Use this to prove that if $C := \sup_n E|X_n|^r < \infty$ then $EM^r \leq C(r/(r-1))^r$.
(C) Give an example of an L^1-bounded martingale $\{X_n\}_{n\geq 0}$ for which $EM = \infty$.

Problem 4. Let D be a bounded, connected, open subset of $\mathbb{R}^d$ with smooth boundary ∂D, and let $U_1, U_2, \cdots$ be independent, identically distributed random vectors such that each U_i is uniformly distributed on the unit sphere S^{d-1} in $\mathbb{R}^d$. Define a random process $\{Z_n\}_{n\geq 0}$ on D as follows: $Z_0 = z \in D$ is an arbitrary starting point, and

$$Z_{n+1} = Z_n + R_n U_{n+1} \quad \text{where } R_n = \frac{1}{2}\text{distance}(Z_n, \partial D).$$

(Thus, conditional on $Z_0, Z_1, \cdots, Z_n$, the random vector Z_{n+1} is uniformly distributed on the sphere centered at Z_n of radius half the current distance to the boundary of D.) Prove that with probability one the sequence Z_n converges to a (random) point $Z_\infty \in \partial D$.

HINT: (a) Since D is bounded, each of the coordinate variables is bounded on D. (b) For each coordinate $i = 1, 2, \ldots, d$ the ith coordinate function u_i is *harmonic* for the Markov chain, that is, the sequence of real random variables $\{u_i(Z_n)\}_{n\geq 0}$ is a *martingale* relative to the natural filtration.

NOTES: (a) This is not at all obvious or easy to prove without using martingale theory. (b) The distribution F_z of the point $\lim Z_n$ is known as the *harmonic measure* of ∂D as seen from z. It can be shown that F_z has a smooth density $p(z, w)$ with respect to the surface area measure on ∂D; the function $p(z, w)$ is the *Poisson kernel* of the domain D.