Problem 1. Let $\mu_n$ and $\mu$ be Borel probability measures on $\mathbb{R}$. For every $\epsilon > 0$ denote by $\nu_{\epsilon}$ the Gaussian distribution with mean 0 and variance $\epsilon^2$, that is,

$$\nu_{\epsilon}(B) = \frac{1}{\sqrt{2\pi\epsilon}} \int_B e^{-x^2/2\epsilon^2} \, dx.$$ 

Prove that if for every $\epsilon > 0$ the sequence $\mu_n \ast \nu_{\epsilon}$ converges weakly to $\mu \ast \nu_{\epsilon}$ then the sequence $\mu_n$ converges weakly to $\mu$.

Problem 2. This problem is meant to show that weak convergence and almost sure convergence are very different things. Let $X_1, X_2, \ldots$ be independent, identically distributed standard normal random variables, and let $S_n$ be the $n$th partial sum of the sequence. Prove that with probability 1 the sequence $S_n/\sqrt{n}$ is dense in $\mathbb{R}$. HINT: The Kolmogorov 0–1 Law might be of some help.

Problem 3. Discrete Cauchy Distribution. Let $S_n = (U_n, V_n)$ be a simple random walk on the even two-dimensional integer lattice $\mathbb{Z}^2$, that is, the increments $\xi_n = S_n - S_{n-1}$ are independent, identically distributed random vectors with common distribution

$$P(\xi_n = (1, 1)) = P(\xi_n = (-1, 1)) = P(\xi_n = (-1, -1)) = P(\xi_n = (1, -1)) = \frac{1}{4}.$$ 

For each $m = 1, 2, \ldots$ define $T_m$ to be the first time $n$ that $U_n = m$, and let $\varphi(\theta) = E e^{i\theta V_1}$ be the characteristic function of the $y$–coordinate at the time $T_1$.

(A) What is the distribution of $T_1$?
(B) Find an explicit formula for $\varphi(\theta)$.
(C) Show that $E e^{i\theta V_m} = \varphi(\theta)^m$.
(D) Show that $V_m/m \Rightarrow$ Cauchy distribution.

HINTS: For (A), review section 4.2 of the Lecture Notes. For (B), condition on the first step of the random walk to obtain a functional equation for $\varphi(\theta)$. Alternatively, use (A). For (C), show that the Fourier transforms converge.

Problem 4. Let $X_1, X_2, \ldots$ be independent, identically distributed Rademacher $\frac{1}{2}$ random variables (that is, random variables that take the values $\pm 1$ with probability $\frac{1}{2}$ each), and let $S_n = \sum_{i=1}^n X_i$. Use the Inversion Formula for Fourier series to show that for any $x \in \mathbb{R}$ the limit

$$\lim_{n \to \infty} \sqrt{n} P\{S_{2n} = 2\lfloor \sqrt{n} x \rfloor\} := \psi(x)$$
exists, and if possible evaluate $\psi(x)$. Here $\lfloor \cdot \rfloor$ denotes the greatest integer function. HINT: First show that

$$P\{S_{2n} = 2k\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\cos \theta)^{2n} \cos(2k\theta) \, d\theta = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} (\cos \theta)^{2n} \cos(2k\theta) \, d\theta.$$  

(You will need the fact that $2k$ and $2n$ are even for this.) Then make the substitution $t = \sqrt{n}\theta$ and use the dominated convergence theorem.

**Bonus Problem** *(not to be turned in).*

In class we showed that there is an even, $C^\infty$ probability density $f$ with support $[-1, 1]$. In this problem you will use the density $f$ to construct another probability density $g$ all of whose moments are finite that is not uniquely determined by its moments. Let $Y, Z$ be independent random variables both with density $f$.

(A) Show that $Y - Z$ has a $C^\infty$ probability density $h$ with support $[-2, 2]$ such that the Fourier transform $\hat{h}$ is nonnegative and satisfies

$$\int_{\mathbb{R}} |\theta|^k \hat{h}(\theta) \, d\theta < \infty$$

for every integer $k \geq 0$. HINT: For the second assertion you will need the fact that $h$ is $C^\infty$ and has compact support.

(B) Part (A) implies that there is a normalizing constant $0 < C < \infty$ such that $g(\theta) := \hat{h}(\theta)/C$ is a probability density on $\mathbb{R}$. Show that the Fourier transform $\hat{g}$ of $g$ is proportional to $h$.

(C) Fix a constant $A > 4$, and define $\psi(\theta) = g(\theta)(1 + \cos \theta A)$. Show that $\psi$ is a probability density on $\mathbb{R}$ whose Fourier transform

$$\hat{\psi}(x) := \int_{\mathbb{R}} e^{ix\theta} \psi(\theta) \, d\theta$$

satisfies

$$\hat{\psi}(x) = \hat{g}(x) \quad \text{for} \ -2 \leq x \leq 2;$$

$$= \hat{g}(x-A)/2 \quad \text{for} \ A-2 \leq x \leq A+2;$$

$$= \hat{g}(x+A)/2 \quad \text{for} \ -A-2 \leq x \leq -A+2;$$

$$= 0 \quad \text{otherwise.}$$

HINT: Since $\psi(\theta) \geq 0$, to prove that it is a probability density it suffices to show that $\hat{\psi}(0) = 1$. This will follow by verifying the equations above, because $\hat{g}(0) = 1$. 
(D) Show that the densities $g$ and $\psi$ have the same moments.