STATISTICS 381: MEASURE-THEORETIC PROBABILITY I
HOMEWORK ASSIGNMENT 8
DUE MARCH 13, 2019

Problem 1. Let μ_n and μ be Borel probability measures on $\mathbb{R}$. For every $\varepsilon > 0$ denote by ν_ϵ the Gaussian distribution with mean 0 and variance ε^2, that is,

$$\nu_\varepsilon(B) = \frac{1}{\sqrt{2\pi}\varepsilon}\int_B e^{-x^2/2\varepsilon^2}\,dx.$$

Prove that if for every $\varepsilon > 0$ the sequence $\mu_n * \nu_\varepsilon$ converges weakly to $\mu * \nu_\varepsilon$ then the sequence μ_n converges weakly to μ.

Problem 2. This problem is meant to show that weak convergence and almost sure convergence are very different things. Let $X_1, X_2, ..$ be independent, identically distributed standard normal random variables, and let S_n be the nth partial sum of the sequence. Prove that with probability 1 the sequence $S_n/\sqrt{n}$ is dense in $\mathbb{R}$. HINT: The Kolmogorov 0–1 Law might be of some help.

Problem 3. *Discrete Cauchy Distribution.* Let $S_n = (U_n, V_n)$ be a simple random walk on the *even* two-dimensional integer lattice $\mathbb{Z}^2$, that is, the increments $\xi_n = S_n - S_{n-1}$ are independent, identically distributed random vectors with common distribution

$$P(\xi_n = (1,1)) = P(\xi_n = (-1,1)) = P(\xi_n = (-1,1)) = P(\xi_n = (-1,-1)) = \frac{1}{4}.$$

For each $m = 1, 2, \dots$ define T_m to be the first time n that $U_n = m$, and let $\varphi(\theta) = E e^{i\theta V_{T_1}}$ be the characteristic function of the $y-$coordinate at the time T_1.

(A) What is the distribution of T_1?
(B) Find an explicit formula for $\varphi(\theta)$.
(C) Show that $E e^{i\theta V_{T_m}} = \varphi(\theta)^m$.
(D) Show that $V_{T_m}/m \Longrightarrow$ Cauchy distribution.

HINTS: For (A), review section 4.2 of the Lecture Notes. For (B), condition on the first step of the random walk to obtain a functional equation for $\varphi(\theta)$. Alternatively, use (A). For (C), show that the Fourier transforms converge.

Problem 4. Let $X_1, X_2, \cdots$ be independent, identically distributed Rademacher $\frac{1}{2}$ random variables (that is, random variables that take the values ± 1 with probability $\frac{1}{2}$ each), and let $S_n = \sum_{i=1}^n X_i$. Use the Inversion Formula for Fourier series to show that for any $x \in \mathbb{R}$ the limit

$$\lim_{n\to\infty} \sqrt{n} P\{S_{2n} = 2[\sqrt{n}x]\} := \psi(x)$$

exists, and if possible evaluate $\psi(x)$. Here $[\cdot]$ denotes the greatest integer function. HINT: First show that

$$\begin{aligned} P\{S_{2n}=2k\} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\cos\theta)^{2n}\cos(2k\theta)\,d\theta \\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} (\cos\theta)^{2n}\cos(2k\theta)\,d\theta. \end{aligned}$$

(You will need the fact that $2k$ and $2n$ are even for this.) Then make the substitution $t = \sqrt{n}\theta$ and use the dominated convergence theorem.

Bonus Problem *(not to be turned in).*

In class we showed that there is an even, C^∞ probability density f with support $[-1,1]$. In this problem you will use the density f to construct another probability density g all of whose moments are finite that is not uniquely determined by its moments. Let Y, Z be independent random variables both with density f.

(A) Show that $Y-Z$ has a C^∞ probability density h with support $[-2,2]$ such that the Fourier transform $\hat{h}$ is nonnegative and satisfies

$$\int_{\mathbb{R}} |\theta|^k \hat{h}(\theta)\,d\theta < \infty$$

for every integer $k \geq 0$. HINT: For the second assertion you will need the fact that h is C^∞ and has compact support.

(B) Part (A) implies that there is a normalizing constant $0 < C < \infty$ such that $g(\theta) := \hat{h}(\theta)/C$ is a probability density on $\mathbb{R}$. Show that the Fourier transform $\hat{g}$ of g is proportional to h.

(C) Fix a constant $A > 4$, and define $\psi(\theta) = g(\theta)(1+\cos\theta A)$. Show that ψ is a probability density on $\mathbb{R}$ whose Fourier transform

$$\hat{\psi}(x) := \int_{\mathbb{R}} e^{i\theta x}\psi(\theta)\,d\theta$$

satisfies

$$\begin{aligned} \hat{\psi}(x) &= \hat{g}(x) \quad \text{for } -2 \leq x \leq 2; \\ &= \hat{g}(x-A)/2 \quad \text{for } A-2 \leq x \leq A+2; \\ &= \hat{g}(x+A)/2 \quad \text{for } -A-2 \leq x \leq -A+2; \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

HINT: Since $\psi(\theta) \geq 0$, to prove that it is a probability density it suffices to show that $\hat{\psi}(0) = 1$. This will follow by verifying the equations above, because $\hat{g}(0) = 1$.

(D) Show that the densities g and ψ have the same moments.