Problem 1. (Tightness).

(A) Show that a sequence of Borel probability measures $\mu_n$ on $\mathbb{R}$ is tight if and only if
$$\lim_{\sigma \to 0} \inf_{n \geq 1} \int e^{-\sigma^2 x^2/2} d\mu_n(x) = 1.$$ 

(B) Show that a sequence of Borel probability measures on $\mathbb{R}$ is tight if and only if there exists a continuous, nonnegative, increasing function $f : [0, \infty) \to [0, \infty)$ such that
$$\lim_{x \to \infty} f(x) = +\infty \quad \text{and} \quad \sup_{n \geq 1} \int f(|x|) d\mu_n(x) < \infty.$$ 

Problem 2. Let $X_1, X_2, \ldots$ be independent, identically distributed standard normal random variables, and let $S_n$ be the $n$th partial sum of the sequence. Prove that with probability 1 the sequence $S_n / \sqrt{n}$ is dense in $\mathbb{R}$.

Problem 3. Discrete Cauchy Distribution. Let $S_n = (U_n, V_n)$ be a simple random walk on the two-dimensional integer lattice $\mathbb{Z}^2$, that is, the increments $\xi_n = S_n - S_{n-1}$ are independent, identically distributed random vectors with common distribution
$$P(\xi_n = (1, 0)) = P(\xi_n = (-1, 0)) = P(\xi_n = (0, 1)) = P(\xi_n = (0, -1)) = \frac{1}{4}.$$ 

For each $m = 1, 2, \ldots$ define $T_m$ to be the first time $n$ that $U_n = m$, and let $\varphi(\theta) = E e^{i\theta V_{T_1}}$ be the characteristic function of the $y-$coordinate at the time $T_1$.

(A) Explain (briefly) why $P(T_m < \infty) = 1$.

(B) Find an explicit formula for $\varphi(\theta)$.

(C) Show that $E e^{i\theta V_{T_m}} = \varphi(\theta)^m$.

(D) Show that $V_{T_m} / m \Rightarrow$ Cauchy distribution.

HINT: For (B), condition on the first step of the random walk to obtain a functional equation for $\varphi(\theta)$. Alternatively, mimic the proof of the third Wald identity to show that if $\cos \theta + \cosh \alpha = 2$ and $\alpha > 0$ then
$$E e^{i\theta V_{T_1} + \alpha} = 1.$$ 

Problem 4. Continuation of Problem 3. Use a variation of the Laplace method to show that for some constant $C > 0$ (which you should evaluate if possible)
(A) For each fixed \( k \in \mathbb{Z} \),
\[
P(V_{T_m} = k) \sim \frac{C}{m} \quad \text{as} \quad m \to \infty.
\]

(B) For any fixed \( \alpha \in \mathbb{R} \),
\[
P\{ V_{T_m} = \lfloor m \alpha \rfloor \} \sim \frac{C}{m} \frac{1}{1 + \pi^2 \alpha^2 / 9}.
\]

**Note:** The result of (B) shows that the random walk \((V_{T_m})_{m \geq 0}\) has a very peculiar property: for every \( \alpha = 0, 1, 2, \ldots \), the expected number of visits to 0 by the random walk \(V_{T_m} - \alpha m\) is *infinite*, and consequently, each of these random walks is *recurrent*.

**Problem 5.** *(Optional, but interesting)* In class we showed that there is an even, \(C^\infty\) probability density \(f\) with support \([-1,1]\). In this problem you will use the density \(f\) to construct another probability density \(g\) all of whose moments are finite that is not uniquely determined by its moments. Let \(Y, Z\) be independent random variables both with density \(f\).

(A) Show that \(Y - Z\) has a \(C^\infty\) probability density \(h\) with support \([-2,2]\) such that the Fourier transform \(\hat{h}\) is nonnegative and satisfies
\[
\int_{\mathbb{R}} |\theta|^k \hat{h}(\theta) \, d\theta < \infty
\]
for every integer \(k \geq 0\). **Hint:** For the second assertion you will need the fact that \(h\) is \(C^\infty\) and has compact support.

(B) Part (A) implies that there is a normalizing constant \(0 < C < \infty\) such that \(g(\theta) := \hat{h}(\theta)/C\) is a probability density on \(\mathbb{R}\). Show that the Fourier transform \(\hat{g}\) of \(g\) is proportional to \(h\).

(C) Fix a constant \(A > 4\), and define \(\psi(\theta) = g(\theta)(1 + \cos \theta A)\). Show that \(\psi\) is a probability density on \(\mathbb{R}\) whose Fourier transform
\[
\hat{\psi}(x) := \int_{\mathbb{R}} e^{i\theta x} \psi(\theta) \, d\theta
\]
satisfies
\[
\hat{\psi}(x) = \hat{g}(x) \quad \text{for} \quad -2 \leq x \leq 2;
\]
\[
= \hat{g}(x - A)/2 \quad \text{for} \quad A - 2 \leq x \leq A + 2;
\]
\[
= \hat{g}(x + A)/2 \quad \text{for} \quad -A - 2 \leq x \leq -A + 2;
\]
\[
= 0 \quad \text{otherwise}.
\]
**Hint:** Since \(\psi(\theta) \geq 0\), to prove that it is a probability density it suffices to show that \(\hat{\psi}(0) = 1\). This will follow by verifying the equations above, because \(\hat{g}(0) = 1\).

(D) Show that the densities \(g\) and \(\psi\) have the same moments.