STATISTICS 381: MEASURE-THEORETIC PROBABILITY I
HOMEWORK ASSIGNMENT 6
DUE FEBRUARY 25, 2019

In these problems all random variables are assumed to be defined on a fixed probability space $(\Omega, \mathscr{F}, P)$.

Problem 1. Quickies. Let $X_1, X_2, \dots$ be any sequence of independent, identically distributed real-valued random variables.

(A) Show that if $E|X_1| = \infty$ then $P\{\limsup_{n\to\infty} |X_n|/n = \infty\} = 1$.

(B) Show that if $E|X_1| < \infty$ then with probability 1,

$$\lim_{n\to\infty} \frac{1}{n} \max_{1\le k\le n} |X_k| = 0.$$

Problem 2. L^2 **SLLN.** This problem outlines an alternative proof of the L^2 Strong Law of Large Numbers. This proof does not rely on either Kronecker's Lemma or Khintchine's theorem (Theorem 4.25 in the Notes) on convergence of random series.

Assume that $X_1, X_2, \cdots$ are independent, identically distributed random variables such that $EX_1 = 0$ and $\sigma^2 := EX_1^2 < \infty$. Let $S_n = \sum_{i=1}^n X_i$ be the nth partial sum.

(A) Use Chebyshev's inequality and the (easy) Borel-Cantelli Lemma to prove that

$$P\{\lim_{m\to\infty} S_{m^2}/m^2 = 0\} = 1.$$

HINT: It suffices to show that for every $\varepsilon > 0$

$$P\{|S_{m^2}| \ge \varepsilon m^2 \text{ infinitely often}\} = 0.$$

(B) Use the Maximal Inequality and the (easy) Borel-Cantelli Lemma to prove that for any $\varepsilon > 0$,

$$P\{\max_{m^2+1\le n\le (m+1)^2} |S_n - S_{m^2}| \ge \varepsilon m^2 \text{ infinitely often}\} = 0.$$

(C) Deduce that $P\{\lim_{n\to\infty} S_n/n = 0\} = 1$.

Problem 3. Subadditivity. A sequence a_n of real numbers is said to be *subadditive* if for every pair $m, n \in \mathbb{N}$,

$$a_{m+n} \le a_m + a_n.$$

(A) Prove *Fekete's Lemma:* For any subadditive sequence a_n of real numbers,

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\ge 1} \frac{a_n}{n} \ge -\infty.$$

(B) Example: Let $X_1, X_2, \cdots$ be independent, identically distributed random variables with partial sums $S_n = \sum_{i=1}^n X_i$. Verify that for any real $\alpha \geq 0$ the sequence $-\log P\{S_n \geq n\alpha\}$ is subadditive, and conclude that

$$\lim_{n\to\infty} P\{S_n \geq n\alpha\}^{1/n} = e^{-\psi(\alpha)}$$

exists. Show also that the function ψ is convex on the interval where it is finite. NOTE: The function ψ is known as the *large deviations rate function.*

(C) Let $(\mathscr{Y}, \mathscr{G})$ be a measurable space. Say that a sequence of measurable functions

$$w_m : \mathscr{Y}^m \to \mathbb{R}$$

is *subadditive* if for any pair $m, n \in \mathbb{N}$ and any elements $y_1, y_2, \cdots \mathscr{Y}$,

$$w_{m+n}(y_1, y_2, \cdots, y_{m+n}) \leq w_m(y_1, y_2, \cdots, y_m) + w_n(y_{m+1}, y_{m+2}, \cdots, y_{m+n}).$$

Let w_m be a subadditive sequence of functions, and let $Y_1, Y_2, \cdots$ be independent, identically distributed random variables taking values in the set $\mathscr{Y}$ such that $E\, w_1^+(Y_1) < \infty$. (Here w_1^+ denotes the positive part of the function w_1.) Show that (i) the sequence of real numbers $E\, w_m(Y_1, Y_2, \cdots, Y_m)$ is subadditive; and (ii) with probability one,

$$\limsup_{m\to\infty} \frac{w_m(Y_1, Y_2, \cdots, Y_m)}{m} \leq \inf_{m\to\infty} \frac{E\, w_m(Y_1, Y_2, \cdots, Y_m)}{m}.$$

HINTS: Use subadditivity to show that for any fixed integer $K \geq 1$, any $1 \leq L \leq K$, and any $n = 1, 2, \cdots$,

$$w_{nK+L}(Y_1, Y_2, \cdots, Y_{nK+L}) \leq \sum_{i=0}^{n-1} w_K(Y_{iK+1}, \cdots, Y_{(i+1)K}) + w_L(Y_{nK+1}, \cdots, Y_{nK+L})$$

and

$$w_L^+(y_1, y_2, \cdots, y_L) \leq \sum_{i=1}^{L} w_1^+(y_i).$$

NOTE: It is also true that with probability 1,

$$\lim_{m\to\infty} \frac{w_m(Y_1, Y_2, \cdots, Y_m)}{m} = \inf_{m\to\infty} \frac{E\, w_m(Y_1, Y_2, \cdots, Y_m)}{m}.$$

This is a special case of Kingman's *subadditive ergodic theorem.* The proof is more difficult than that of the one-sided bound given above.

Example (not to be turned in): Let $X_1, X_2, \cdots$ be independent, identically distributed random vectors taking values in the $d-$dimensional integer lattice $\mathbb{Z}^d$. Define

$$S_n = \sum_{i=1}^n X_i \quad \text{and}$$
$$R_n = \#\{S_1, S_2, \cdots, S_n\},$$

where # denotes cardinality. (Thus, R_n is the number of distinct sites visited by the random walk S_m in its first n steps.) Use the (special case of the) subadditive ergodic theorem to show that with probability 1,

$$\lim_{n\to\infty} \frac{R_n}{n} = P\{S_m \neq 0 \ \forall\, m \geq 1\}.$$