In these problems all random variables are assumed to be defined on a fixed probability space \((\Omega, \mathcal{F}, P)\).

**Problem 1.** Prove that if \(X_1, X_2, \ldots\) are independent, identically distributed Bernoulli–\(p\) random variables then there is no subsequence \(n_k\) of the positive integers such that

\[
P\{\lim_{k \to \infty} X_{n_k} \text{ exists}\} = 1.
\]

Try to do this in two different ways:

(a) using Kolmogorov’s 0–1 Law; and

(b) using the 2nd (hard) Borel-Cantelli Lemma.

**Problem 2.** Let \(X_1, X_2, \ldots\) be independent, identically distributed random variables with common mean \(EX_i = 0\), and let \(S_n = \sum_{i=1}^{n} X_i\). Assume that the random variables \(X_i\) are uniformly bounded, i.e., that there exists \(K < \infty\) such that \(|X_i| \leq K\). Use the first Wald identity together with the strong Markov property to show that with probability 1,

\[
\limsup_{n \to \infty} S_n = +\infty \quad \text{and} \quad \liminf_{n \to \infty} S_n = -\infty.
\]

**Problem 3.** Let \(X_1, X_2, \ldots\) be independent (but not necessarily identically distributed) random variables with \(EX_i = 0\) and \(EX_i^2 = \sigma^2_i < \infty\). Set \(S_n = \sum_{i=1}^{n} X_i\). Let \(T\) be a finite stopping time for the sequence \(X_1, X_2, \ldots\) (i.e., for each \(n = 0, 1, 2, \ldots\) the event \(\{T = n\}\) is measurable with respect to \(\sigma(X_i)_{i \leq n}\)).

(A) Prove that if \(T\) is bounded (i.e., \(T \leq m\) for some finite integer \(m\)) then

\[
(1) \quad E S_T^2 = E \sum_{i=1}^{T} \sigma_i^2.
\]

(B) Prove that if \(\sum_{i=1}^{\infty} \sigma_i^2 < \infty\) then the identity \(1)\) holds for every finite stopping time \(T\).

(C) Show by example that the identity \(1)\) need not hold if \(T\) is not bounded, even if \(T\) has finite expectation \(ET < \infty\).

**Problem 4.** Let \(\{S_n\}_{n \geq 0}\) be simple random walk started at \(S_0 = 0\), that is, \(S_n = \sum_{i=1}^{n} X_i\) where \(X_1, X_2, \ldots\) are independent, identically distributed Rademacher–\(\frac{1}{2}\) random variables. Fix
integers $-A < 0 < B$ and let $T = T_{[-A,B]}$ be the first time that the random walk visits either $-A$ or $+B$. Use the third Wald identity to evaluate the generating functions

$$
\psi_+(s) := E s^T 1\{S_T = +B\} \quad \text{and} \quad \psi_-(s) := E s^T 1\{S_T = -A\}.
$$

Use your formulas to deduce as much as you can about the distribution of $T$.

**Problem 5.** Let $X_1, X_2, \ldots$ be independent, identically distributed, integer-valued random variables from a distribution with a geometric right tail, that is, for some $0 < \alpha < 1$ and $C > 0$,

$$
P\{X_n = k\} = C \alpha^k \quad \text{for all} \quad k = 1, 2, 3, \ldots.
$$

(Note: There are no restrictions on the probabilities $P\{X_n = k\}$ for negative integers $k$.)

Assume further that $E|X_n| < \infty$ and that $\mu := EX_n > 0$. Fix an integer $A \geq 1$, and let

$$
T := \min\{n \geq 1 : S_n > A\} \quad \text{where} \quad S_n = \sum_{i=1}^{n} X_i.
$$

(A) Show that $S_T - A$ has the geometric distribution

$$
P\{S_T - A = k\} = (1 - \alpha) \alpha^{k-1} \quad \text{for all} \quad k = 1, 2, 3, \ldots.
$$

(B) Evaluate $ET$.

**Bonus Problem: Wiener-Hopf factorization.** This problem shows how an extension of Wald’s third identity can be used in combination with polynomial algebra to solve a random walk problem that cannot be solved by elementary combinatorial methods. Let $X_1, X_2, \ldots$ be independent, identically distributed random variables that take values in the set $V = \{-L, -L+1, \ldots, M\}$ where $L, M$ are positive integers. For each $k \in V$ let

$$
p_k = P\{X_i = k\},
$$

and assume that $p_{-L} > 0$ and $p_M > 0$. Set $S_n = \sum_{i=1}^{n} X_i$, and define the ladder times and ladder heights

$$
T_+ = \min\{n \geq 1 : S_n \geq 1\},
$$

$$
T_- = \min\{n \geq 1 : S_n \leq 0\},
$$

$$
S_+ = S_{T_+},
$$

$$
S_- = S_{T_-}.
$$

Let

$$
Q(z) = \sum_{k=-L}^{M} p_k z^k = E z^{X_i}
$$

be the probability generating function of $X_i$. This is finite for every $z \in \mathbb{C}$. 
(A) Show that for any bounded stopping time $\tau$ and any $z \in \mathbb{C}$,
$$E\left(\frac{z^{S_\tau}}{Q(z)^\tau}\right) = 1.$$

(B) Let $\beta \in \mathbb{C}$ be a root of the equation $Q(\beta) = 1$. Show that
$$E\beta^{S_+} = 1 \quad \text{if} \quad |\beta| \geq 1 \quad \text{and} \quad E\beta^{S_-} = 1 \quad \text{if} \quad |\beta| \leq 1.$$

(C) Show that $Ez^{S_+}$ is a polynomial of degree $M$ in $z$, and that $Ez^{-S_-}$ is a polynomial of degree $L$ in $z$. **Note:** Here you must make use of the assumption that $p_M > 0$ and $p_- > 0$.

(D) Show that the equation $Q(\beta) = 1$ has exactly $M + L$ complex roots (counted according to multiplicity), and check that $\beta = 1$ is a root.

(E) Show further that there are exactly $M$ roots of $Q(\beta) = 1$ such that $|\beta| \geq 1$, and that if these roots are listed as $\beta_1, \beta_2, \ldots, \beta_M$ then
$$Ez^{S_+} = 1 + C_+ \prod_{j=1}^{M}(z - \beta_j) \quad \text{where} \quad C_+ = (-1)^M \prod_{i=1}^{M-1} \beta_i.$$

Show also that there are exactly $L$ roots of $Q(\beta) = 1$ such that $|\beta| \leq 1$, and that if these roots are listed as $\alpha_1, \alpha_2, \ldots, \alpha_L$ then

$$Ez^{-S_-} = 1 + C_- \prod_{j=1}^{L}(z - \alpha^{-1}_j) \quad \text{where} \quad C_- = p_-.$$

**Hint:** To prove the formula for $C_+$, use the fact that the constant term of the polynomial $Ez^{S_+}$ is 0. To prove the formula for $C_-$, observe that the only way that the event $S_- = -L$ can occur is for the very first step to be $-L$.

(F) Example: Suppose that $L = 1$, $M = 2$, and that
$$p_- = p_1 = p_2 = 1/3.$$

Find the distributions of $S_-$ and $S_+$. 