Problem 1. Fat Cantor Sets. This problem is prompted by a mistake that several people made on the midterm. Is it true that for every Borel set $B \subset [0,1]$ and every $\epsilon > 0$ there is an open set $U \subset B$ such that $\lambda(B \setminus U) < \epsilon$? The answer is no. This problem outlines a construction of a closed set $B$ that cannot be approximated from within by open subsets.

Recall that the Cantor set $C$ is obtained by an inductive process where at each step one removes the middle third of each remaining interval from the previous step; what is left is $C$. Thus,

$$C = \bigcap_{n=0}^{\infty} C_n$$

where $C_0 = [0,1]$ and each $C_{n+1}$ is obtained by removing the (open) middle third of each interval in $C_n$. Induction shows that $C_n$ is the union of $2^n$ non-overlapping closed intervals, each of length $3^{-n}$.

(A) Prove that the Lebesgue measure of $C$ is 0.

Now let $B$ be the set obtained in analogous fashion to the Cantor set, but at the $n$th stage removing the middle $1/5^n$ instead of the middle third. Thus,

$$B = \bigcap_{n=0}^{\infty} B_n$$

where $B_0 = [0,1]$ and each $B_{n+1}$ is obtained by removing the (open) middle $1/5^n$th portion of each interval in $B_n$.

(B) Show that the Lebesgue measure of $B$ is positive (and calculate it!).

(C) Show that no nonempty open set is contained in $B$.

Problem 2. Mixing. A measure-preserving transformation $T$ of a probability space $(\Omega, \mathcal{F}, P)$ is said to be mixing if for any two bounded random variables $f, g : \Omega \to \mathbb{R},$

$$\lim_{n \to \infty} E f(g \circ T^n) = (Ef)(Eg).$$

(A) Show that if $T$ is mixing then $T$ is ergodic.

(B) Show that if $\mathcal{A}$ is an algebra such that $\mathcal{F} = \sigma(\mathcal{A})$ then $T$ is mixing if for all $A, B \in \mathcal{A},$

$$\lim_{n \to \infty} E 1_A(1_B \circ T^n) = P(A)P(B).$$
(C) Let $T$ be the shift on $(\mathbb{R}^\infty, \mathcal{B}^\infty, \nu^\infty)$ (See notes for definitions. The probability measure $\nu^\infty$ is the product measure; under $\nu^\infty$ the coordinate variables are i.i.d. with distribution $\nu$.) Show that $T$ is mixing, and therefore ergodic. Conclude that the strong law of large numbers is a special case of the ergodic theorem.

**Problem 3. Multiparameter Ergodic Theorem.** Let $S$ and $T$ be ergodic, measure-preserving transformations of a probability space $(\Omega, \mathcal{F}, \mu)$, and let $Y : \Omega \to \mathbb{R}$ be a bounded random variable such that $E Y = 0$. Using the strategy outlined below, prove that

\[
\lim_{m,n \to \infty} \frac{1}{mn} \sum_{i=0}^{m} \sum_{j=0}^{n} Y \circ T^i \circ S^j = 0 \quad \text{almost surely.}
\]

(A) Fix $\epsilon > 0$ small. Use Birkhoff’s Theorem to show that for sufficiently large $m(\epsilon)$,

\[
E \sup_{m \geq m(\epsilon)} \left| \frac{1}{m} \sum_{i=1}^{m} Y \circ S^i \right| < \epsilon^2.
\]

(B) Deduce from (A) that $\mu(B_\epsilon) \leq \epsilon$, where $B_\epsilon$ is the event

\[
B_\epsilon = \left\{ \sup_{m \geq m(\epsilon)} \left| \frac{1}{m} \sum_{i=1}^{m} Y \circ S^i \right| \geq \epsilon \right\}.
\]

(C) Use Birkhoff’s Theorem a second time to show that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n} 1_{B_\epsilon} \circ T^j \leq \epsilon.
\]

and use this to deduce (1).

**Problem 4.** Use the result of problem 3 to prove the following theorem: If $X_1, X_2, \ldots$ is a sequence of independent, identically distributed random variables and if $h : \mathbb{R}^2 \to \mathbb{R}$ is a bounded, Borel measurable function then

\[
\lim_{n \to \infty} \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} h(X_i, X_j) = E h(X_1, X_2).
\]

**Problem 5.** **Some $L^2$ Theory.** Not to be turned in. Do this problem if you are not already familiar with the basic facts about orthogonal projections in $L^2$—spaces. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $L^2$ be the space of (equivalence classes of) square-integrable real-valued functions on $\Omega$. For any two elements $f, g \in L^2$ define

\[
\langle f, g \rangle = \int f g \, d\mu \quad \text{and} \quad \|f\|_2^2 = \langle f, f \rangle.
\]

Say that $f, g$ are orthogonal is $\langle f, g \rangle = 0$. Say that $f$ is orthogonal to a subset $A \subset L^2$ if $\langle f, g \rangle = 0$ for every $g \in A$. 
(A) Prove the parallelogram law \( \|f + g\|_2^2 + \|f - g\|_2^2 = 2\|f\|_2^2 + 2\|g\|_2^2 \).

(B) A set \( C \subset L^2 \) is convex if for any \( f, g \in C \) and any \( t \in [0, 1] \) the convex combination \( tf + (1 - t)g \in C \). Show that if \( C \) is a nonempty, closed, convex subset of \( L^2 \) then \( C \) has a unique element of minimum \( L^2 \)-norm.

**HINT:** Let \( \delta = \inf_{f \in C} \|f\|_2 \). Use the parallelogram law to show that if \( f, g \in C \) then
\[
\|f - g\|_2^2 \leq 2\|f\|_2^2 + 2\|g\|_2^2 - 4\delta^2.
\]
Use this to show that if \( C \) has an element of minimum norm, then it is unique. Then use it to show that if \( f_n \in C \) is a sequence such that
\[
\lim_{n \to \infty} \|f_n\|_2 = \delta
\]
then the sequence \( f_n \) is Cauchy.

(C) Now prove that if \( V \subset L^2 \) is a closed linear subspace and \( f \notin V \) then there is a unique element \( g \in V \) (called the orthogonal projection of \( f \) on \( V \)) such that
\[
\|f - g\|_2 = \inf_{h \in V} \|f - h\|_2.
\]
Finally, show that \( f - g \) is orthogonal to \( V \).