STATISTICS 381: MEASURE-THEORETIC PROBABILITY I
HOMEWORK ASSIGNMENT 4
DUE FEBRUARY 4, 2019

Problem 1. Let $\{X_\theta\}_{\theta\in\Theta}$ be a collection of L^1 random variables defined on a probability space $(\Omega, \mathscr{F}, P)$. Prove that the collection $\{X_\theta\}_{\theta\in\Theta}$ is uniformly integrable if and only if the following two conditions hold:

(i) $\sup_{\theta\in\Theta} E|X_\theta| < \infty$; and
(ii) for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any event A of probability $\leq \delta$ and every $\theta \in \Theta$,
$$E|X_\theta|\mathbf{1}_A \leq \varepsilon.$$

HINT: For one of the two directions you might find it useful to know the *Markov inequality* (which you should prove if you don't already know it): for any nonnegative random variable Y and every real $\alpha > 0$
$$\alpha P\{|Y| \geq \alpha\} \leq E\,Y.$$

Problem 2. If there is a function $\varphi : [0,\infty) \to [0,\infty]$ such that $\varphi(x)/x \to \infty$ as $x \to \infty$ and if
$$\sup_{\theta\in\Theta} E\,\varphi(|X_\theta|) < \infty$$
then the collection $\{X_\theta\}_{\theta\in\Theta}$ is uniformly integrable. Thus, for example, if the second moments $E\,X_\theta^2$ are uniformly bounded then the collection $\{X_\theta\}_{\theta\in\Theta}$ is uniformly integrable.

Problem 3. Let $\{X_\theta\}_{\theta\in\Theta}$ and $\{Y_\theta\}_{\theta\in\Theta}$ be indexed collections of random variables defined on a probability space $(\Omega, \mathscr{F}, P)$.

(A) Show that if there is a nonnegative, integrable random variable X such that $|X_\theta| \leq X$ for every $\theta \in \Theta$, then the collection $\{X_\theta\}_{\theta\in\Theta}$ is uniformly integrable.

(B) Show that if the collection $\{Y_\theta\}_{\theta\in\Theta}$ is uniformly integrable, and if $|X_\theta| \leq |Y_\theta|$ for every $\theta \in \Theta$, then the collection $\{X_\theta\}_{\theta\in\Theta}$ is uniformly integrable.

Problem 4. Let $\Omega = \mathbb{R}$, $\mathscr{F} = \mathscr{B}$, and $\mu =$ Lebesgue measure.

(A) Show that for every function $f \in L^1$ and every $\varepsilon > 0$ there exists a step function $g \in L^1$ such that $\|f - g\|_1 < \varepsilon$. HINT: First show that this is true for every function f of the form $f = \mathbf{1}_B$, where B is a Borel set with finite measure. NOTE: A *step function* $g : \mathbb{R} \to \mathbb{R}$ is a function of the form
$$g = \sum_{i=1}^{m} a_i \mathbf{1}_{J_i}$$
where $J_1, J_2, \ldots, J_m$ are *intervals* and $a_1, a_2, \ldots, a_m$ are *scalars*.

(B) Show that for every function $f \in L^1$ and every $\varepsilon > 0$ there exists a *continuous* function $g : \mathbb{R} \to \mathbb{R}$ with compact support such that $\|f - g\|_1 < \varepsilon$.

(C) For any function $f : \mathbb{R} \to \mathbb{R}$ and any $x \in \mathbb{R}$, define the $x-$*translate* of f to be the function $\tau_x f : \mathbb{R} \to \mathbb{R}$ defined by

$$(\tau_x f)(y) = f(y - x).$$

Prove that for any function $f \in L^1$,

$$\lim_{\varepsilon \to 0} \|\tau_\varepsilon f - f\|_1 = 0$$

HINT: First show that this is true for *continuous* functions f with compact support.

Problem 5. True or false? – There is a Borel set $B \subset [0,1]$ such that for every interval $J \subset [0,1]$,

$$\mu(B \cap J) = \frac{1}{2}\mu(J).$$

(Here μ is Lebesgue measure.)