Problem 1. Let \( \{X_\theta\}_{\theta \in \Theta} \) be a collection of \( L^1 \) random variables defined on a probability space \((\Omega, \mathcal{F}, P)\). Prove that the collection \( \{X_\theta\}_{\theta \in \Theta} \) is uniformly integrable if and only if the following two conditions hold:

(i) \( \sup_{\theta \in \Theta} E|X_\theta| < \infty \); and

(ii) for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for any event \( A \) of probability \( \leq \delta \) and every \( \theta \in \Theta \),
\[
E|X_\theta|1_A \leq \epsilon.
\]

HINT: For one of the two directions you might find it useful to know the Markov inequality (which you should prove if you don’t already know it): for any nonnegative random variable \( Y \) and every real \( \alpha > 0 \)
\[
\alpha P\{|Y| \geq \alpha\} \leq EY.
\]

Problem 2. If there is a function \( \varphi : [0, \infty) \to [0, \infty] \) such that \( \varphi(x)/x \to \infty \) as \( x \to \infty \) and if
\[
\sup_{\theta \in \Theta} E \varphi(|X_\theta|) < \infty
\]
then the collection \( \{X_\theta\}_{\theta \in \Theta} \) is uniformly integrable. Thus, for example, if the second moments \( E X_\theta^2 \) are uniformly bounded then the collection \( \{X_\theta\}_{\theta \in \Theta} \) is uniformly integrable.

Problem 3. Let \( \{X_\theta\}_{\theta \in \Theta} \) and \( \{Y_\theta\}_{\theta \in \Theta} \) be indexed collections of random variables defined on a probability space \((\Omega, \mathcal{F}, P)\).

(A) Show that if there is a nonnegative, integrable random variable \( X \) such that \( |X_\theta| \leq X \) for every \( \theta \in \Theta \), then the collection \( \{X_\theta\}_{\theta \in \Theta} \) is uniformly integrable.

(B) Show that if the collection \( \{Y_\theta\}_{\theta \in \Theta} \) is uniformly integrable, and if \( |X_\theta| \leq |Y_\theta| \) for every \( \theta \in \Theta \), then the collection \( \{X_\theta\}_{\theta \in \Theta} \) is uniformly integrable.

Problem 4. Let \( \Omega = \mathbb{R} \), \( \mathcal{F} = \mathcal{B} \), and \( \mu = \) Lebesgue measure.

(A) Show that for every function \( f \in L^1 \) and every \( \epsilon > 0 \) there exists a step function \( g \in L^1 \) such that \( \|f - g\|_1 < \epsilon \). HINT: First show that this is true for every function \( f \) of the form \( f = 1_B \), where \( B \) is a Borel set with finite measure. NOTE: A step function \( g : \mathbb{R} \to \mathbb{R} \) is a function of the form
\[
g = \sum_{i=1}^m a_i 1_{J_i}
\]
where \( J_1, J_2, \ldots, J_m \) are intervals and \( a_1, a_2, \ldots, a_m \) are scalars.
(B) Show that for every function $f \in L^1$ and every $\epsilon > 0$ there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ with compact support such that $\|f - g\|_1 < \epsilon$.

(C) For any function $f : \mathbb{R} \to \mathbb{R}$ and any $x \in \mathbb{R}$, define the $x$–translate of $f$ to be the function $\tau_x f : \mathbb{R} \to \mathbb{R}$ defined by

$$(\tau_x f)(y) = f(y - x).$$

Prove that for any function $f \in L^1$,

$$\lim_{\epsilon \to 0} \|\tau_{\epsilon} f - f\|_1 = 0$$

HINT: First show that this is true for continuous functions $f$ with compact support.

Problem 5. True or false? – There is a Borel set $B \subset [0, 1]$ such that for every interval $J \subset [0, 1]$,

$$\mu(B \cap J) = \frac{1}{2} \mu(J).$$

(Here $\mu$ is Lebesgue measure.)