In these problems all random variables are assumed to be defined on a fixed probability space \((\Omega, \mathcal{F}, P)\).

**Problem 1.** Let \(X_1, X_2, \ldots\) be independent, identically distributed random variables with common mean \(E X_i = 0\). Assume that these random variables are uniformly bounded, i.e., that there exists \(K < \infty\) such that \(|X_i| \leq K\). Show that the sequence of partial sums
\[
S_n = \sum_{i=1}^{n} X_i
\]
visits the interval \([-K, K]\) infinitely often, with probability 1.

**Problem 2.** Let \(X_1, X_2, \ldots\) be independent (but not necessarily identically distributed) random variables with \(E X_i = 0\) and \(E X_i^2 = \sigma_i^2 < \infty\). Set \(S_n = \sum_{i=1}^{n} X_i\). Let \(T\) be a finite stopping time for the sequence \(X_1, X_2, \ldots\) (i.e., for each \(n = 0, 1, 2, \ldots\) the event \(\{T = n\}\) is measurable with respect to \(\sigma(X_i)_{i \leq n}\)).

(A) Prove that if \(T\) is bounded (i.e., \(T \leq m\) for some finite integer \(m\)) then
\[
E S_T^2 = E \sum_{i=1}^{T} \sigma_i^2.
\]

(B) Prove that if \(\sum_{i=1}^{\infty} \sigma_i^2 < \infty\) then the identity (1) holds for every finite stopping time \(T\).

(C) Show by example that the identity (1) need not hold if \(T\) is not bounded, even if \(T\) has finite expectation \(E T < \infty\).

**Problem 3.** Let \(X_1, X_2, \ldots\) be any sequence of real-valued random variables.

(A) Show that there are scalars \(a_n > 0\) such that the series \(\sum_{n=1}^{\infty} a_n X_n\) converges almost surely to a finite limit \(S\). HINT: Start from the observation made in class that if \((y_n)_{n \geq 1}\) is any sequence of real numbers such that \(|y_n - y_{n+1}| < 2^{-n}\) eventually then \(\lim_{n \to \infty} y_n\) exists.

(B) Assume now that the random variables \(X_1, X_2, \ldots\) are independent. Show that if the series \(\sum_{n=1}^{\infty} X_n\) converges almost surely to a finite constant then the random variables \(X_n\) are themselves constants. HINT: First show that if two random variables \(X, Y\) are independent then \(X + Y\) is (almost surely) constant if and only if \(X\) and \(Y\) are themselves constant.
(C) Assume that the random variables $X_1, X_2, \ldots$ are independent. Show that if there is a sequence of scalars $a_n \to 0$ such that $\lim_{m \to \infty} a_n \sum_{i=1}^{m} X_i$ exists and is finite with probability one then the limit is a constant random variable.

**Problem 4.** Let $\{S_n\}_{n \geq 0}$ be simple random walk started at $S_0 = 0$, that is, $S_n = \sum_{i=1}^{n} X_i$ where $X_1, X_2, \ldots$ are independent, identically distributed Rademacher $\pm 1$ random variables. Fix integers $-A < 0 < B$ and let $T = T_{[-A,B]}$ be the first time that the random walk visits either $-A$ or $+B$. Use the third Wald identity to evaluate the generating functions

$$
\psi_+(s) := E s^T 1\{S_T = +B\} \quad \text{and} \quad \psi_-(s) := E s^T 1\{S_T = -A\}.
$$

Use your formulas to deduce as much as you can about the distribution of $T$.

**Problem 5.** A *skip-free* random walk on $\mathbb{Z}$ is a sequence of random variables $S_n = \sum_{i=1}^{n} X_i$ such that the increments $X_i$ are independent, identically distributed integer-valued random variables that satisfy $X_i \leq 1$. Thus, the random walk $S_n$ can make big downward jumps, but can only move upward in steps of size 1.

(A) Let $x_1, x_2, \ldots, x_n$ be integers all $\leq 1$ such that $\sum_{i=1}^{n} x_i = 1$. Show that there is a unique *circulant permutation* $\pi$ of the integers $\{1, 2, \ldots, n\}$ such that

$$
\sum_{i=1}^{k} x_{\pi(i)} \leq 0 \quad \text{for each} \quad k \leq n - 1.
$$

**NOTE:** The circulant permutations of $1, 2, 3, 4$ are

- 1234
- 2341
- 3412
- 4123.

In general, a circulant permutation is a permutation whose adjacency matrix is a circulant matrix.

(B) Now let $S_n = \sum_{i=1}^{n} X_i$ be a skip-free random walk, and let $\tau = \min\{n : S_n = +1\}$ (or $\tau = \infty$ if there is no such $n$). Show that for any $n \geq 1$,

$$
P\{\tau = n\} = \frac{1}{n} P\{S_n = 1\}.
$$

Note that in the special case of simple random walk, this implies that

$$
P\{\tau = 2n + 1\} = \frac{1}{2n + 1} \binom{2n + 1}{n + 1} \frac{1}{2^{2n+1}}.
$$
The following problem is optional, and not to be turned in. It shows how an extension of Wald’s third identity can be used in combination with polynomial algebra to solve a random walk problem that cannot be solved by elementary combinatorial methods.

**Problem 6. Wiener-Hopf factorization.** Let $X_1, X_2, \ldots$ be independent, identically distributed random variables that take values in the set $V = \{-L, -L + 1, \ldots, M\}$ where $L, M$ are positive integers. For each $k \in V$ let

$$p_k = P\{X_i = k\},$$

and assume that $p_{-L} > 0$ and $p_M > 0$. Set $S_n = \sum_{i=1}^n X_i$, and define the ladder times and ladder heights

$$T_+ = \min\{n \geq 1 : S_n \geq 1\},$$
$$T_- = \min\{n \geq 1 : S_n \leq 0\},$$
$$S_+ = S_{T_+},$$
$$S_- = S_{T_-}.$$

Let

$$Q(z) = \sum_{k=-L}^{M} p_k z^k = E z^{X_i}$$

be the probability generating function of $X_i$. This is finite for every $z \in \mathbb{C}$.

(A) Let $\beta \in \mathbb{C}$ be a root of the equation $Q(\beta) = 1$. Show that

$$E \beta^{S_+} = 1 \quad \text{if} \quad |\beta| \leq 1 \quad \text{and}$$
$$E \beta^{S_-} = 1 \quad \text{if} \quad |\beta| \geq 1.$$

(B) Show that $E z^{S_+}$ is a polynomial of degree $M$ in $z$, and that $E z^{-S_-}$ is a polynomial of degree $L$ in $z$. **Note:** Here you must make use of the assumption that $p_M > 0$ and $p_{-L} > 0$.

(C) Show that the equation $Q(\beta) = 1$ has exactly $M + L$ complex roots (counted according to multiplicity), and check that $\beta = 1$ is a root.

(D) Show further that there are exactly $M$ roots of $Q(\beta) = 1$ such that $|\beta| \geq 1$, and that if these roots are listed as $\beta_1, \beta_2, \ldots, \beta_M$ then

$$E z^{S_+} = 1 + C_+ \prod_{j=1}^{M} (z - \beta_j) \quad \text{where} \quad C_+ = (-1)^{M-1} \prod_{i=1}^{M-1} \beta_i.$$

Show also that there are exactly $L$ roots of $Q(\beta) = 1$ such that $|\beta| \leq 1$, and that if these roots are listed as $\alpha_1, \alpha_2, \ldots, \alpha_L$ then

(2) $$E z^{-S_-} = 1 + C_- \prod_{j=1}^{L} (z - \alpha_j^{-1}) \quad \text{where} \quad C_- = p_{-L}.$$
Hint: To prove the formula for \( C_+ \), use the fact that the constant term of the polynomial \( Ez^n \) is 0. To prove the formula for \( C_- \), observe that the only way that the event \( S_- = -L \) can occur is for the very first step to be \(-L\).

(E) Numerical example: Suppose that \( X_n = U_n - V_n \) where \( U_n, V_n \) are independent and uniformly distributed on \{1,2,3,4,5,6\}. Find \( P\{S_\pm = k\} \) for each \( k = 1,2,\ldots,6 \) by numerically solving for the (complex) roots of \( Q(\beta) = 1 \) and then using the results of part (D).