STATISTICS 381: MEASURE-THEORETIC PROBABILITY I
HOMEWORK ASSIGNMENT 3
DUE JANUARY 28, 2019

Problem 1. Let $(X_n)_{n\in\mathbb{N}}$ and X be real-valued random variables all defined on a common probability space $(\Omega, \mathscr{F}, P)$. Say that the sequence X_n converges to X *in probability* if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} P\{|X_n - X| > \varepsilon\} = 0.$$

Say that the sequence X_n converges to X *almost surely* if

$$P\{\lim_{n\to\infty} X_n = X\} = 1.$$

(A) Prove that if X_n converges to X almost surely then X_n converges to X in probability.

(B) The converse is not true: in fact, it is possible that X_n converges to X in probability but $P\{\lim_{n\to\infty} X_n = X\} = 0$. Find an example of this.

Problem 2. Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $A_1, A_2, \ldots, A_n \in \mathscr{F}$ be any events. Set $A = \cup_{i=1}^n A_i$.

(A) Show that $\mathbf{1}_A = 1 - \prod_{i=1}^n (1 - \mathbf{1}_{A_i})$.

(B) Use this to prove the *inclusion-exclusion* formula

$$P(A) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} P(\cap_{i=1}^n A_i).$$

(C) Let F_n be a random permutation of the set $[n] := \{1, 2, \cdots, n\}$, that is, F_n is a random object taking values in the set $\mathscr{S}_n$ of all bijections $f : [n] \to [n]$ such that for any fixed element $f \in \mathscr{S}_n$,

$$P\{F_n = f\} = \frac{1}{n!}.$$

Let Y_n be the number of fixed points of F_n, i.e.,

$$Y_n = \sum_{i=1}^n \mathbf{1}_{\{F_n(i)=i\}}.$$

Show that

$$\lim_{n\to\infty} P\{Y_n = 0\}$$

exists, and find its value.

Bonus Problem. *Weierstrass' Approximation Theorem.* Weierstrass' theorem states that any continuous function on the unit interval can be arbitrarily well-approximated by a polynomial in the supremum norm, that is, if $f : [0,1] \to \mathbb{R}$ is continuous, then for any $\varepsilon > 0$ there exists a polynomial $p(x)$ such that

$$\sup_{x\in[0,1]} |f(x) - p(x)| < \varepsilon.$$

This problem outlines a probabilistic proof of this fact. Let $U_1, U_2, \ldots$ be independent, identically distributed random variables defined on a probability space $(\Omega, \mathscr{F}, P)$ such that each U_i has the uniform distribution on $[0,1]$.[1] For each $n \geq 1$ and $x \in [0,1]$, define

$$p_n(x) = Ef\left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{[0,x]}(U_i)\right).$$

(A) Show that $p_n(x)$ is a polynomial of degree n.

(B) Show that for each $x \in [0,1]$,

$$\lim_{n\to\infty} p_n(x) = f(x).$$

(C) Now show that the convergence in (B) holds *uniformly* in x. NOTE: You can use the fact that a continuous function on $[0,1]$ is uniformly continuous and has a finite max and min.

[1] We proved in class that such a sequence can be defined on Lebesgue space.