Problem 1. Let \((X_n)_{n \in \mathbb{N}}\) and \(X\) be real-valued random variables all defined on a common probability space \((\Omega, \mathcal{F}, P)\). Say that the sequence \(X_n\) converges to \(X\) in probability if for every \(\varepsilon > 0\),
\[
\lim_{n \to \infty} P\{|X_n - X| > \varepsilon\} = 0.
\]
Say that the sequence \(X_n\) converges to \(X\) almost surely if
\[
P\{\lim_{n \to \infty} X_n = X\} = 1.
\]

(A) Prove that if \(X_n\) converges to \(X\) almost surely then \(X_n\) converges to \(X\) in probability.

(B) The converse is not true: in fact, it is possible that \(X_n\) converges to \(X\) in probability but \(P\{\lim_{n \to \infty} X_n = X\} = 0\). Find an example of this.

Problem 2. Let \((\Omega, \mathcal{F}, P)\) be a probability space and let \(A_1, A_2, \ldots, A_n \in \mathcal{F}\) be any events. Set \(A = \bigcup_{i=1}^n A_i\).

(A) Show that \(1_A = 1 - \prod_{i=1}^n (1 - 1_{A_i})\).

(B) Use this to prove the inclusion-exclusion formula
\[
P(A) = \sum_{i} P(A_i) - \sum_{i < j} P(A_i \cap A_j) + \sum_{i < j < k} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} P(\bigcap_{i=1}^n A_i).
\]

(C) Let \(F_n\) be a random permutation of the set \([n] := \{1, 2, \ldots, n\}\), that is, \(F_n\) is a random object taking values in the set \(\mathcal{S}_n\) of all bijections \(f : [n] \to [n]\) such that for any fixed element \(f \in \mathcal{S}_n\),
\[
P\{F_n = f\} = \frac{1}{n!}.
\]
Let \(Y_n\) be the number of fixed points of \(F_n\), i.e.,
\[
Y_n = \sum_{i=1}^n 1_{\{F_n(i) = i\}}.
\]
Show that
\[
\lim_{n \to \infty} P\{Y_n = 0\}
\]
exists, and find its value.
**Bonus Problem.** *Weierstrass’ Approximation Theorem.* Weierstrass’ theorem states that any continuous function on the unit interval can be arbitrarily well-approximated by a polynomial in the supremum norm, that is, if \( f : [0, 1] \to \mathbb{R} \) is continuous, then for any \( \varepsilon > 0 \) there exists a polynomial \( p(x) \) such that

\[
\sup_{x \in [0,1]} |f(x) - p(x)| < \varepsilon.
\]

This problem outlines a probabilistic proof of this fact. Let \( U_1, U_2, \ldots \) be independent, identically distributed random variables defined on a probability space \((\Omega, \mathcal{F}, P)\) such that each \( U_i \) has the uniform distribution on \([0, 1]\). For each \( n \geq 1 \) and \( x \in [0, 1] \), define

\[
p_n(x) = E f \left( \frac{1}{n} \sum_{i=1}^{n} 1_{[0,x]}(U_i) \right).
\]

(A) Show that \( p_n(x) \) is a polynomial of degree \( n \).

(B) Show that for each \( x \in [0, 1] \),

\[
\lim_{n \to \infty} p_n(x) = f(x).
\]

(C) Now show that the convergence in (B) holds uniformly in \( x \). **Note:** You can use the fact that a continuous function on \([0, 1]\) is uniformly continuous and has a finite max and min.

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\(^1\)We proved in class that such a sequence can be defined on Lebesgue space.