In these problems all random variables and events are assumed to be defined on a fixed probability space \((\Omega, \mathcal{F}, P)\), and all algebras and \(\sigma\)-algebras are assumed to be contained in \(\mathcal{F}\).

**Problem 1. Littlewood’s Approximation Principle.** Problem 4 (d) on HW assignment 1 essentially states that if \(\mathcal{A}\) is an algebra then for every event \(F \in \sigma(\mathcal{A})\) and every \(\epsilon > 0\) there is an event \(A \in \mathcal{A}\) such that 
\[
P(A \Delta F) < \epsilon.
\]
This is a version of one of Littlewood’s Three Principles.

(A) Use Littlewood’s Principle and the \(\pi-\lambda\) Lemma to prove that if \(\{\mathcal{A}_n\}_{n \geq 1}\) is a sequence of independent algebras then \(\{\sigma(\mathcal{A}_n)\}_{n \geq 1}\) is a sequence of independent \(\sigma\)-algebras.

(B) Use the result of (A) to prove the following fact. If \(\{\epsilon_{m,n}\}_{m,n \geq 1}\) is an infinite double array of independent Bernoulli-\(\frac{1}{2}\) random variables then the random variables 
\[
U_m := \sum_{n=1}^{\infty} \frac{\epsilon_{m,n}}{2^n}
\]
are independent, identically distributed, all with the uniform distribution on \([0,1]\). Conclude that Lebesgue space (i.e., the probability space \(([0,1], \mathcal{B}, m))\) supports a sequence of independent random variables \(U_1, U_2, \ldots\) all with the uniform distribution on \([0,1]\).

**Problem 2.** Let \(X_1, X_2, \ldots, X_n\) be independent, identically distributed random variables defined on a probability space \((\Omega, \mathcal{F}, P)\). Show that for every Borel subset \(B\) of \(\mathbb{R}^n\) and every permutation \(\pi\) of the integers \([n]\), 
\[
P\{\{X_1, X_2, \ldots, X_n\} \in B\} = P\{\{X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(n)}\} \in B\}.
\]

**Note:** In general, a set of random variables \(X_1, X_2, \ldots, X_n\) such that this equation holds for every Borel set \(B\) is said to be *exchangeable*. Thus, this problem shows that independent, identically distributed random variables are exchangeable.

**Problem 3.** A \(\sigma\)-algebra \(\mathcal{G}\) on a set \(\Omega\) is said to be *countably generated* if there is a countable collection \(\{A_n\}_{n \geq 1}\) of events \(A_n \in \mathcal{G}\) such that \(\mathcal{G} = \sigma(\{A_n\}_{n \geq 1})\), equivalently, if there is a sequence \(X_n\) of Bernoulli random variables (indicator functions), all measurable with respect to \(\mathcal{G}\), such that \(\mathcal{G} = \sigma(X_n)_{n \geq 1}\). Prove that \(\mathcal{G}\) is countably generated if and only if there is a real random variable \(Y : \Omega \to [0,1]\) (measurable with respect to \(\mathcal{G}\)) such that \(\mathcal{G} = \sigma(Y)\).
NOTE: Recall that the $\sigma$–algebra $\sigma(Y_\theta)_{\theta \in \Theta}$ generated by a collection of random variables (measurable functions) $Y_\theta$ is the smallest $\sigma$–algebra containing all events of the form 

$$\{Y_\theta \in B\} \quad \text{where} \quad B \in \mathcal{B}_R.$$ 

HINT: If $\{X_n\}_{n \geq 1}$ is a sequence of Bernoulli random variables then the series 

$$Y = \sum_{n=1}^{\infty} 2X_n / 3^n$$ 

defines a real random variable $Y$. Show that if $\mathcal{G} = \sigma(X_n)_{n \geq 1}$ then $\mathcal{G} = \sigma(Y)$.

**Problem 4. Nearest Neighbor Random Walk on $\mathbb{Z}$.** Let $X_1, X_2, \ldots$ be independent, identically distributed random variables on a probability space $(\Omega, \mathcal{F}, P)$ with common distribution 

$$P\{X_i = +1\} = p = 1 - P\{X_i = -1\},$$ 

and let $S_n = \sum_{i=1}^{n} X_i$. The sequence $\{S_n\}_{n \geq 0}$ is called nearest neighbor random walk on $\mathbb{Z}$.

(A) Prove that $P\{\limsup |S_n| = \infty\} = 1$.

(B) Prove that if $p > 1/2$ then $P\{\lim S_n = +\infty\} = 1$.

(C) Prove that if $p = 1/2$ then 

$$P\{\limsup S_n = +\infty\} = P\{\liminf S_n = -\infty\} = 1.$$ 

Observe that (C) implies that if $p = 1/2$ then with probability one the sequence $S_n$ visits every integer infinitely often.

HINT: For part (A) you must show that for every integer $m \geq 1$ the event $\{|S_n| \leq m \text{ for all } n \geq 0\}$ has probability zero. For part (C) you should use Kolmogorov’s 0-1 Law.

**Problem 5.** Let $X_1, X_2, \ldots$ be a sequence of independent random variables such that

(i) for each $n$ there is a finite set $F_n$ such that $P(X_n \in F_n) = 1$; and 

(ii) $S := \lim_{n \to \infty} \sum_{i=1}^{n} X_i$ exists and is finite almost surely.

Prove that either there is a countable set $A$ such that $P(S \in A) = 1$ or there is no $\alpha \in \mathbb{R}$ such that $P(S = \alpha) > 0$. 