STATISTICS 381: MEASURE-THEORETIC PROBABILITY I
HOMEWORK ASSIGNMENT 2
DUE WEDNESDAY JANUARY 23, 2019

Problem 1. Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $\mathscr{A}$ be an algebra of subsets of Ω such that $\mathscr{F} = \sigma(\mathscr{A})$ (that is, $\mathscr{F}$ is the smallest $\sigma-$algebra containing $\mathscr{A}$). Prove that for every event $F \in \mathscr{F}$ and every real number $\varepsilon > 0$ there is an event $A \in \mathscr{A}$ such that

$$P(A \Delta F) < \varepsilon.$$

NOTE: This is one of *Littlewood's 3 Principles.*

Problem 2. Prove that for every Borel set $B \in \mathscr{B}_{\mathbb{R}}$ and every real number x the set

$$B + x := \{y \in \mathbb{R} : y - x \in B\}$$

is a Borel set.

Problem 3. Let $X_1, X_2, \cdots$ be a sequence of (real) random variables defined on a probability space $(\Omega, \mathscr{F}, P)$. Show that the function

$$Y := \limsup_{n\to\infty} X_n$$

is a random variable (that is, Y is a measurable transformation). NOTE: The function Y might take the value $+\infty$, so *measurability* means measurability with respect to the Borel $\sigma-$algebra on $\mathbb{R} \cup \{\infty\}$.

Problem 4. A $\sigma-$algebra $\mathscr{G}$ on a set Ω is said to be *countably generated* if there is a countable collection $\{A_n\}_{n\geq 1}$ of events $A_n \in \mathscr{G}$ such that $\mathscr{G} = \sigma(\{A_n\}_{n\geq 1})$, equivalently, if there is a sequence X_n of Bernoulli random variables (indicator functions), all measurable with respect to $\mathscr{G}$, such that $\mathscr{G} = \sigma(X_n)_{n\geq 1}$. Prove that $\mathscr{G}$ is countably generated if and only if there is a real random variable $Y : \Omega \to [0, 1]$ (measurable with respect to $\mathscr{G}$) such that $\mathscr{G} = \sigma(Y)$.

NOTE: By definition, the $\sigma-$algebra $\sigma(Y_\theta)_{\theta\in\Theta}$ generated by a collection of random variables Y_θ is the smallest $\sigma-$algebra containing all events of the form

$$\{Y_\theta \in B\} \quad \text{where} \quad B \in \mathscr{B}_{\mathbb{R}}.$$

HINT: Show that if $\{X_n\}_{n\geq 1}$ is a sequence of Bernoulli random variables then the series

$$Y = \sum_{n=1}^{\infty} 2X_n/3^n$$

defines a real random variable Y. Then show that if $\mathscr{G} = \sigma(X_n)_{n\geq 1}$ then $\mathscr{G} = \sigma(Y)$.

Problem 5. (A) Prove that if $\{\mathscr{A}_n\}_{n\geq 1}$ is a sequence of independent *algebras* then $\{\sigma(\mathscr{A}_n)\}_{n\geq 1}$ is a sequence of independent $\sigma-$*algebras*. HINT: Littlewood's Principle (Problem 5) might be of some use here.

(B) Use the result of (A) to prove the following fact. If $\{\varepsilon_{m,n}\}_{m,n\geq 1}$ is an infinite double array of independent Bernoulli—$\frac{1}{2}$ random variables then the random variables

$$U_m := \sum_{n=1}^{\infty} \varepsilon_{m,n}/2^n$$

are independent, identically distributed, all with the uniform distribution on $[0,1]$. Conclude that *Lebesgue space* (i.e., the probability space $([0,1], \mathscr{B}, m)$) supports a sequence of independent random variables $U_1, U_2, \dots$ all with the uniform distribution on $[0,1]$.

Problem 6. Nearest Neighbor Random Walk on $\mathbb{Z}$. Let $X_1, X_2, \dots$ be independent, identically distributed random variables on a probability space $(\Omega, \mathscr{F}, P)$ with common distribution

$$P\{X_i = +1\} = p = 1 - P\{X_i = -1\},$$

and let $S_n = \sum_{i=1}^n X_i$. The sequence $\{S_n\}_{n\geq 0}$ is called *nearest neighbor random walk* on $\mathbb{Z}$.

(A) Prove that $P\{\limsup |S_n| = \infty\} = 1$.
(B) Prove that if $p > 1/2$ then $P\{\lim S_n = +\infty\} = 1$.
(C) Prove that if $p = 1/2$ then

$$P\{\limsup S_n = +\infty\} = P\{\liminf S_n = -\infty\} = 1.$$

Observe that (C) implies that if $p = 1/2$ then with probability one the sequence S_n visits every integer infinitely often.

HINT: For part (A) you must show that for every integer $m \geq 1$ the event $\{|S_n| \leq m \text{ for all } n \geq 0\}$ has probability zero.

Bonus Problem. Let $X_1, X_2, \dots$ be a sequence of independent random variables such that

(i) for each n there is a finite set F_n such that $P(X_n \in F_n) = 1$; and
(ii) $S := \lim_{n\to\infty} \sum_{i=1}^n X_i$ exists and is finite almost surely.

Prove that *either* there is a *countable* set A such that $P(S \in A) = 1$ *or* there is no $\alpha \in \mathbb{R}$ such that $P(S = \alpha) > 0$.