Problem 1. Sequence Space. Let $\Omega = \{0, 1\}^\mathbb{N}$ be the space of infinite sequences of 0s and 1s. Define a metric on $\Omega$ by setting $d(x, y) = 2^{-n(x, y)}$ where $n(x, y)$ is defined to be the maximum $n$ such that $x_i = y_i$ for all $i \leq n$.

(a) Show that this is in fact a metric.
(b) Show that the metric space $(\Omega, d)$ is compact.
(c) Show that the Borel $\sigma$–algebra on $(\Omega, d)$ is the $\sigma$–algebra generated by the cylinder sets.
(d) Show that the set of sequences $x_n$ satisfying $\lim_{n \to \infty} n^{-1} \sum_{i=1}^{n} x_i = 1/3$ is a Borel set.

NOTE: The Borel $\sigma$–algebra is by definition the $\sigma$–algebra generated by the open subsets of $\Omega$.

Problem 2. Prove that a $\sigma$–algebra cannot be countably infinite – it must either be finite or uncountable.

Problem 3. Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ be a nested sequence of $\sigma$–algebras on a set $\Omega$.

(A) Show that $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is a field.
(B) Show by example that $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ need not be a $\sigma$–algebra.

Problem 4. For any two events $A, B$ define $d(A, B) = P(A \Delta B)$, where $A \Delta B$ is the symmetric difference between $A, B$. Say that two events $A, A'$ are almost surely equal if $d(A, A') = 0$.

(a) Check that almost surely equal is an equivalence relation.
(b) Show that $d$ is a metric on the set $\mathcal{F}^*$ of equivalence classes of events.
(c) Show that the metric space $(\mathcal{F}^*, d)$ is complete.
(d) Suppose that $\mathcal{F} = \sigma(\mathcal{A})$ where $\mathcal{A}$ is an algebra. Let $\mathcal{A}^*$ be the set of equivalence classes that contain elements of $\mathcal{A}$. Show that $\mathcal{A}^*$ is dense in $(\mathcal{F}^*, d)$.

HINT: For (c), extract a subsequence and use Borel-Cantelli.

Problem 5. A probability measure $P$ on a $\sigma$–algebra $\mathcal{F}$ is said to be non-atomic if for every $F \in \mathcal{F}$ such that $P(F) > 0$ there exists $G \in \mathcal{F}$ such that $G \subset F$ and $0 < P(G) < P(F)$.

(A) Show that Lebesgue measure on the unit interval is non-atomic.
(B) Show that if $P$ is non-atomic then for every $F \in \mathcal{F}$ of positive measure $P(F) > 0$ and every real $x \in (0, P(F))$ there exists $G \in \mathcal{F}$ such that $G \subset F$ and $P(G) = x$.
(C) Show that if $P$ is non-atomic then there is a collection of events $\{G_t\}_{t \in [0,1]}$ indexed by the real numbers between 0 and 1 such that

(i) if $s \leq t$ then $G_s \subset G_t$, and
(ii) $P(G_t) = t$ for every $t \in [0,1]$.

(D) Conclude that if $P$ is a nonatomic probability measure on a measurable space $(\Omega, \mathcal{F})$ then there is a random variable $T : \Omega \rightarrow [0,1]$ whose distribution is the uniform distribution on $[0,1]$. 

1