STATISTICS 381: MEASURE-THEORETIC PROBABILITY I
HOMEWORK ASSIGNMENT 1
DUE JANUARY 9, 2016

Problem 1. Sequence Space. Let $\Omega = \{0,1\}^{\mathbb{N}}$ be the space of infinite sequences of 0s and 1s. Define a metric on Ω by setting $d(x,y) = 2^{-n(x,y)}$ where $n(x,y)$ is defined to be the maximum n such that $x_i = y_i$ for all $i \le n$.

(a) Show that this is in fact a metric.
(b) Show that the metric space (Ω, d) is compact.
(c) Show that the Borel $\sigma-$algebra on (Ω, d) is the $\sigma-$algebra generated by the cylinder sets.
(d) Show that the set of sequences x_n satisfying $\lim_{n\to\infty} n^{-1}\sum_{i=1}^{n} x_i = 1/3$ is a Borel set.

NOTE: The Borel $\sigma-$algebra is by definition the $\sigma-$algebra generated by the open subsets of Ω.

Problem 2. Prove that a $\sigma-$algebra cannot be countably infinite – it must either be finite or uncountable.

Problem 3. Let $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \cdots$ be a nested sequence of $\sigma-$algebras on a set Ω.

(A) Show that $\cup_{n=1}^{\infty}\mathscr{F}_n$ is a *field*.
(B) Show by example that $\cup_{n=1}^{\infty}\mathscr{F}_n$ need not be a $\sigma-$algebra.

Problem 4. Let $(\Omega, \mathscr{F}, P)$ be a probability space. For any two events $A, B \in \mathscr{F}$ define $d(A,B) = P(A\Delta B)$, where $A\Delta B$ is the symmetric difference between A, B. Say that two events A, A' are *almost surely equal* if $d(A,A') = 0$.

(a) Check that *almost surely equal* is an equivalence relation.
(b) Show that d is a *metric* on the set $\mathscr{F}^*$ of equivalence classes of events.
(c) Show that the metric space $(\mathscr{F}^*, d)$ is *complete*.
(d) Suppose that $\mathscr{F} = \sigma(\mathscr{A})$ where $\mathscr{A}$ is an algebra. Let $\mathscr{A}^*$ be the set of equivalence classes that contain elements of $\mathscr{A}$. Show that $\mathscr{A}^*$ is dense in $(\mathscr{F}^*, d)$.

HINT: For (c), extract a subsequence and use Borel-Cantelli.

Problem 5. A probability measure P on a $\sigma-$algebra $\mathscr{F}$ is said to be *non-atomic* if for every $F \in \mathscr{F}$ such that $P(F) > 0$ there exists $G \in \mathscr{F}$ such that $G \subset F$ and $0 < P(G) < P(F)$.

(A) Show that Lebesgue measure on the unit interval is non-atomic.
(B) Show that if P is non-atomic then for every $F \in \mathscr{F}$ of positive measure $P(F) > 0$ and every real $x \in (0, P(F))$ there exists $G \in \mathscr{F}$ such that $G \subset F$ and $P(G) = x$.
(C) Show that if P is non-atomic then there is a collection of events $\{G_t\}_{t\in[0,1]}$ indexed by the real numbers between 0 and 1 such that

(i) if $s \le t$ then $G_s \subset G_t$, and
(ii) $P(G_t) = t$ for every $t \in [0,1]$.

(D) Conclude that if P is a nonatomic probability measure on a measurable space $(\Omega, \mathscr{F})$ then there is a random variable $T : \Omega \to [0,1]$ whose distribution is the uniform distribution on $[0,1]$.

Problem 6. Define a $d-$dimensional *rectangle* to be a Cartesian product $A_1 \times A_2 \times \cdots \times A_d$, where each A_i is an *interval* of the real line. Let $\mathscr{A}_d$ be the algebra generated by all rectangles. Prove that there is a unique measure $\lambda = \lambda_d$ on $\mathscr{A}_d$ such that for any rectangle,

$$\lambda(A_1 \times A_2 \times \cdots \times A_d) = \prod_{i=1}^{d} |A_i|,$$

where $|A_i|$ is the length of the interval A_i. CLARIFICATION: What must be proved is that λ is countably additive on the *algebra* consisting of finite unions of rectangles. It will then follow immediately from Carathéodory's theorem that the measure extends uniquely to the smallest $\sigma-$algebra containing the rectangles, which is the $d-$dimensional Borel $\sigma-$algebra $\mathscr{B}_d$. This measure is the $d-$dimensional *Lebesgue measure*.