

7 Fourier Transforms

7.1 Preliminaries

The exponential function $e^z = \exp(z)$ is *defined* by the power series⁷ $e^z = \sum_{n=0}^{\infty} z^n / n!$. This series converges absolutely for every complex number z , and uniformly in every disk $|z| \leq R$ of finite radius $0 \leq R < \infty$. The following elementary properties of the exponential function will be taken as *known*.

Properties of the Exponential Function:

- (1) $\exp(z) = \sum_{n=0}^{\infty} z^n / n!$
- (2) $\exp(z + w) = \exp(z) \exp(w)$ for all $z, w \in \mathbb{C}$.
- (3) $\exp(i\theta) = \cos \theta + i \sin \theta$ for all $\theta \in \mathbb{R}$.
- (4) $(d/dz) \exp(z) = \exp(z)$.
- (5) $\exp(\log t) = \log(\exp(t)) = t$ for all $t > 0$.

The infinite series (1) converges absolutely and uniformly in the disk $|z| \leq R$, for any $R < \infty$. The identity (3) is *Euler's formula*; it implies that $|e^{i\theta}| = 1$ for any real θ , and also that the mapping $\theta \mapsto e^{i\theta}$ from \mathbb{R} to the unit circle $\{|\xi| = 1\}$ is 2π -periodic. Euler's formula also implies that every complex number $z = x + iy$ has a *polar* representation $z = |z|e^{i\theta}$, where $|z| = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$.

Exercise 7.1. Prove that for any sequence z_n of complex numbers,

$$\lim_{n \rightarrow \infty} z_n = z \implies \lim_{n \rightarrow \infty} \left(1 + \frac{z_n}{n}\right)^n = e^z. \quad (7.1)$$

Exercise 7.2. The integral of a Borel-measurable complex-valued function $f : \Omega \rightarrow \mathbb{C}$ on a measure space $(\Omega, \mathcal{F}, \mu)$ is defined by

$$\int f \, d\mu = \int \Re f \, d\mu + i \int \Im f \, d\mu.$$

Show that the integral is well-defined for any function f such that the real-valued function $|f|$ is integrable, and that

$$\left| \int f \, d\mu \right| \leq \int |f| \, d\mu. \quad (7.2)$$

7.2 Characteristic Functions and Fourier Transforms

Definition 7.3. Let μ be a finite Borel measure on \mathbb{R} and let $f \in L^1(\mathbb{R})$ be an integrable function (relative to Lebesgue measure). The *Fourier transforms* $\hat{\mu}$ and \hat{f} of μ and f are

⁷The series should be easy for a probabilist or statistician to remember, because for *positive* values of z the equation $e^z = \sum_{n=0}^{\infty} z^n / n!$ is equivalent (after you divide both sides by e^z) to the fact that the Poisson distribution $n \mapsto z^n e^{-z} / n!$ with mean parameter z is actually a probability distribution on the nonnegative integers.

the complex-valued functions

$$\hat{\mu}(\theta) := \int e^{i\theta x} d\mu(x) \quad \text{and} \quad (7.3)$$

$$\hat{f}(\theta) := \int_{\mathbb{R}} e^{i\theta x} f(x) dx. \quad (7.4)$$

The functions $\hat{\mu}$ and \hat{f} are well-defined for all arguments $\theta \in \mathbb{R}$; in fact, the inequality (7.2) implies that they are bounded functions:

$$|\hat{\mu}(\theta)| \leq \hat{\mu}(0) = \mu(\mathbb{R}) \quad \text{and} \quad |\hat{f}(\theta)| \leq \|f\|_1.$$

Moreover, both $\hat{\mu}$ and \hat{f} are *uniformly continuous* on \mathbb{R} , by a routine application of the dominated convergence theorem.

Definition 7.4. The *characteristic function* of a real random variable X is the Fourier transform of its distribution, equivalently, it is the function $\varphi_X : \mathbb{R} \rightarrow \mathbb{C}$ defined by

$$\varphi_X(\theta) = Ee^{i\theta X}. \quad (7.5)$$

Proposition 7.5. If X_1, X_2, \dots are independent integer-valued random variables with partial sum $S_n = \sum_{k=1}^n X_k$ then

$$\varphi_{S_n}(\theta) = \prod_{k=1}^n \varphi_{X_k}(\theta). \quad (7.6)$$

Therefore, in particular, if the random variables X_1, X_2, \dots are independent and identically distributed then

$$\varphi_{S_n}(\theta) = \varphi_{X_1}(\theta)^n. \quad (7.7)$$

This follows immediately from the multiplication rule for expectations of products of independent random variables. There are corresponding product rules for the convolutions of finite Borel measures and L^1 functions: in particular, for any two finite Borel measures μ, ν on \mathbb{R} and any two L^1 functions f, g ,

$$\widehat{\mu * \nu}(\theta) = \hat{\mu}(\theta)\hat{\nu}(\theta) \quad \text{and} \quad (7.8)$$

$$\widehat{f * g}(\theta) = \hat{f}(\theta)\hat{g}(\theta). \quad (7.9)$$

These both follow by routine application of Fubini's theorem.

Proposition 7.6. Let μ be a finite Borel measure on \mathbb{R} with finite k th moment for some integer $k \geq 1$. Then the Fourier transform $\hat{\mu}(\theta)$ has k bounded, continuous derivatives, and these are given by the formula

$$\frac{d^k}{d\theta^k} \hat{\mu}(\theta) = \int (ix)^k e^{i\theta x} d\mu(x). \quad (7.10)$$

Proof. It suffices (by induction and the linearity of the integral) to prove this for the case $k = 1$. For this, take difference quotients and use the dominated convergence theorem. \square

In the special case where μ is a probability measure, the formula (7.10), specialized to $\theta = 0$, shows that the moments of a random variable X can be read off from the derivatives of the characteristic function $\phi_x(\theta)$ at 0:

$$i^k EX^k = \frac{d^k}{d\theta^k} \phi_X(0). \quad (7.11)$$

Proposition 7.7. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a C^∞ function such that f and its first two derivative f' are integrable. Then

$$\sup_{\theta \in \mathbb{R}} |\theta \hat{f}(\theta)| \leq \|f'\|_1 \quad \text{and} \quad \hat{f}(\theta) = \frac{-1}{i\theta} \hat{f}'(\theta). \quad (7.12)$$

Proof. Exercise. (Hint: Integrate by parts. The smoothness hypothesis guarantees that the Fourier transforms of f and all its derivatives decay rapidly at ∞ .) \square

Example 7.8. Let f be the standard normal probability density. The function f is infinitely differentiable, and all of its derivatives are integrable, so (7.12) applies. Now

$$\begin{aligned} \hat{f}'(\theta) &= \int e^{i\theta x} (-xe^{-x^2/2}) dx / \sqrt{2\pi} \\ &= \int xe^{i\theta x} (-e^{-x^2/2}) dx / \sqrt{2\pi} \\ &= \int \left(-i \frac{d}{d\theta} e^{i\theta x} \right) (-e^{-x^2/2}) dx / \sqrt{2\pi} \\ &= i \frac{d}{d\theta} \hat{f}(\theta). \end{aligned}$$

Thus, (7.12) shows that the Fourier transform \hat{f} satisfies the first-order differential equation

$$\hat{f}(\theta) = -\frac{1}{\theta} \frac{d}{d\theta} \hat{f}(\theta).$$

Together with the auxiliary condition $\hat{f}(0) = 1$, this differential equation implies that

$$\hat{f}(\theta) = e^{-\theta^2/2}. \quad (7.13)$$

Example 7.9. The Fourier transform of the double-exponential density $f(x) = \frac{1}{2}e^{-|x|}$ is gotten by a simple integration:

$$\hat{f}(\theta) = \frac{1}{1 + \theta^2}.$$

This is proportional to the *Cauchy density*. Later we will see (by the Fourier inversion theorem) that the reverse is also true: the Fourier transform of the Cauchy density is the double exponential function.

7.3 \mathbb{Z} -Valued Random Variables

If X is an integer-valued random variable then its characteristic function $\varphi_X(\theta)$ is given by the *Fourier series*

$$\varphi_X(\theta) = Ee^{i\theta X} = \sum_{-\infty}^{\infty} P(X = m)e^{i\theta m}. \quad (7.14)$$

This function is 2π -periodic. In this case, the distribution of the random variable X can be recovered from the characteristic function by using the following *orthogonality relations* for the complex exponential functions $e^{im\theta}$.

Proposition 7.10. For $m \in \mathbb{Z}$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{im\theta} d\theta = \delta_{m,0}$$

where $\delta_{m,0}$ is the Kronecker delta function (0 if $m \neq 0$ and 1 if $m = 0$).

This is proved by a trivial integration. Despite the easy proof, the result is of fundamental importance, as it implies that the complex exponential functions $e^{im\theta}$ (also known as the *group characters* for the additive group \mathbb{Z}) are *orthonormal* in $L^2(d\theta/2\pi)$.

Corollary 7.11. For an integer-valued random variable X with characteristic function $\varphi_X(\theta)$ the probability distribution is given by

$$P(X = m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-im\theta} \varphi_X(\theta) d\theta.$$

7.4 Fourier Inversion

We have seen that certain of the important features of a measure are captured by its Fourier transform: for instance, if μ is a probability measure then its moments (when they are finite) can be determined from the derivatives of $\hat{\mu}$ at $\theta = 0$. In this section we will address more systematically the general problem of recovering information about a measure μ from its Fourier transform. Following is a list of the main theorems to be proved.

Theorem 7.12. (*Uniqueness*) Finite Borel measures are uniquely determined by their Fourier transforms, that is, if two Borel measures have the same Fourier transform then they are the same measure.

Theorem 7.12 will follow immediately from the following theorem, which gives an explicit formula showing how the the integral of a bounded, continuous function f against a finite measure μ can be recovered from the Fourier transforms of f and μ .

Theorem 7.13. (*Plancherel-Parseval Formula*) For any finite Borel measure μ and any bounded, continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ with compact support,

$$\int f(x) d\mu(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\theta) \hat{\mu}(-\theta) e^{-\varepsilon^2 \theta^2 / 2} d\theta. \quad (7.15)$$

The hypothesis that f have compact support is needed to guarantee that the Fourier transform \hat{f} is well-defined. The factor $e^{-\varepsilon^2 \theta^2 / 2}$ in the integral is needed because in general the Fourier transforms \hat{f} and $\hat{\mu}$, although bounded, are not integrable. As $\varepsilon \rightarrow 0$ these factors increase to 1, so if the product $\hat{f}(\theta) \hat{\mu}(-\theta)$ happens to be integrable then it will follow, by the dominated convergence theorem, that

$$\int f(x) d\mu(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\theta) \hat{\mu}(-\theta) d\theta. \quad (7.16)$$

Corollary 7.14. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is twice continuously differentiable and has compact support then for every finite Borel measure μ the identity (7.16) holds.

Proof. If f is C^2 and has compact support then its Fourier transform decays at least as fast as C/θ^2 as $|\theta| \rightarrow \infty$. (This can be seen by doing two successive integrations by parts in the integral defining \hat{f} .) Consequently, \hat{f} is integrable, and since the Fourier transform $\hat{\mu}$ of a finite Borel measure is always bounded, it then follows that the product $\hat{f}(\theta) \hat{\mu}(-\theta)$ is integrable. \square

If the measure μ has an integrable Fourier transform $\hat{\mu}$ then the identity (7.16) will hold for every continuous function f with compact support, regardless of smoothness. In this case, even more can be said:

Theorem 7.15. (*Existence of Densities*) If the Fourier transform $\hat{\mu}(\theta)$ of a finite Borel measure is an integrable function (with respect to Lebesgue measure on \mathbb{R}) then the measure μ has a bounded, uniformly continuous density g given by

$$g(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\mu}(\theta) e^{-i\theta x} d\theta. \quad (7.17)$$

Example 7.16. Let μ be the probability measure on \mathbb{R} with the *double-exponential* density $g(x) = e^{-|x|}/2$. A routine integration shows that

$$\hat{\mu}(\theta) (= \hat{g}(\theta)) = \frac{1}{1 + \theta^2}.$$

Consequently, the inversion formula (7.17) implies that if ν is the probability measure with the *Cauchy density* $\pi^{-1}(1 + x^2)^{-1}$ then

$$\hat{\nu}(\theta) = e^{-|\theta|}.$$

This example is interesting (and important) for a number of reasons. First, it shows that the Fourier transform of a probability distribution with infinite first moment need not be differentiable at $\theta = 0$. Second, it shows that if X_1, X_2, \dots, X_n are independent, identically distributed, all with the Cauchy distribution ν , then the sample average S_n/n also has the Cauchy distribution. To see this, observe that by the multiplication rule for characteristic functions of independent random variables, the characteristic function of S_n/n is $e^{-|\theta|}$; hence, by the uniqueness theorem (Theorem 7.12), the distribution of S_n/n must be the same as that of X_1 .

Proofs of the Main Theorems

Fourier analysis on the real line \mathbb{R} is complicated by the fact that Fourier transforms need not be integrable, and so it is often necessary to multiply by “convergence factors”, such as $e^{-\theta^2 \varepsilon^2 / 2}$, to obtain convergent Fourier integrals. The choice of the normal density is useful not only because it has such rapid decrease at ∞ but because *the normal density is its own Fourier transform* (apart from the always annoying factors of 2π): if Z is standard normal, then

$$E e^{i\theta Z} = e^{-\theta^2 / 2}.$$

Thus, if

$$\varphi_\varepsilon(x) := \frac{1}{\sqrt{2\pi\varepsilon}} e^{-x^2 / 2\varepsilon^2} \quad \text{then} \quad \hat{\varphi}_\varepsilon(\theta) = e^{-\varepsilon^2 \theta^2 / 2}.$$

Lemma 7.17. *If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function with compact support, then for every $\varepsilon > 0$*

$$f * \varphi_\varepsilon(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f * \varphi_\varepsilon}(\theta) e^{-i\theta x} d\theta. \quad (7.18)$$

Remark 7.18. This equality asserts that every function of the form $f * \varphi_\varepsilon$ can be recovered from its Fourier transform by *Fourier inversion*. It can be shown that the same is true for any C^2 function with compact support. Since we will not need this fact, we won't prove it.

Proof of Lemma 7.17. Fubini. □

Lemma 7.19. *For any bounded, continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$,*

$$\lim_{\varepsilon \rightarrow 0} f * \varphi_\varepsilon(x) = f(x). \quad (7.19)$$

Proof. Let Z be a random variable defined on some probability space (Ω, \mathcal{F}, P) with standard Gaussian distribution $N(0, 1)$; then

$$f * \varphi_\varepsilon(x) = E f(x + \varepsilon Z).$$

Hence, the convergence (7.19) follows by the dominated convergence theorem. □

Proof of the Plancherel-Parseval Formula (Theorem 7.13). By Lemma 7.19 and the dominated convergence theorem,

$$\int f(x) d\mu(x) = \lim_{\varepsilon \rightarrow 0} \int (f * \varphi_\varepsilon)(x) d\mu(x).$$

But by Lemma 7.17 and Fubini's theorem,

$$\begin{aligned} \int (f * \varphi_\varepsilon)(x) d\mu(x) &= \frac{1}{2\pi} \iint \widehat{f * \varphi_\varepsilon}(\theta) e^{-i\theta x} d\theta d\mu(x) \\ &= \frac{1}{2\pi} \iint \widehat{f * \varphi_\varepsilon}(\theta) e^{-i\theta x} d\mu(x) d\theta \\ &= \frac{1}{2\pi} \int \widehat{f * \varphi_\varepsilon}(\theta) \hat{\mu}(-\theta) d\theta \\ &= \frac{1}{2\pi} \int \hat{f}(\theta) e^{-\varepsilon^2 \theta^2 / 2} \hat{\mu}(-\theta) d\theta. \end{aligned}$$

□

Proof of Theorem 7.15. Assume that the Fourier transform $\hat{\mu}(\theta)$ is an integrable function, and define $g(x)$ by (7.17). To prove that μ has density g it suffices to prove that for any continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ with compact support,

$$\int f(x) d\mu(x) = \int f(x) g(x) dx.$$

The function g is bounded and continuous, by the usual arguments, but *a priori* we do not know that it is integrable. Nevertheless, if f is any bounded continuous function with compact support then $f g$ is integrable, since g is bounded, and furthermore, for any $\varepsilon > 0$ the function $f * \varphi_\varepsilon$ is integrable (Fubini) and so $g(x)(f * \varphi_\varepsilon)(x)$ is also integrable. By definition of g and Fubini, for any $\varepsilon > 0$,

$$\begin{aligned} \int f * \varphi_\varepsilon(x) g(x) dx &= \frac{1}{2\pi} \iint f * \varphi_\varepsilon(x) e^{-i\theta x} \hat{\mu}(\theta) d\theta dx \\ &= \frac{1}{2\pi} \iint f * \varphi_\varepsilon(x) e^{-i\theta x} \hat{\mu}(\theta) dx d\theta \\ &= \frac{1}{2\pi} \int \hat{f}(-\theta) \hat{\varphi}_\varepsilon(-\theta) \hat{\mu}(\theta) d\theta. \end{aligned}$$

As $\varepsilon \rightarrow 0$ the last integral converges to $\int f d\mu$. Therefore, to complete the proof it suffices to show that

$$\lim_{\varepsilon \rightarrow 0} \int f * \varphi_\varepsilon(x) g(x) dx = \int f(x) g(x) dx.$$

But by Fubini, for any $\varepsilon > 0$

$$\begin{aligned}\int f * \varphi_\varepsilon(x) g(x) dx &= \iint f(y) \varphi_\varepsilon(x-y) g(x) dy dx \\ &= \iint f(y) \varphi_\varepsilon(x-y) g(x) dx dy \\ &= \int f(y) (g * \varphi_\varepsilon)(y) dy.\end{aligned}$$

Lemma 7.19 implies that as $\varepsilon \rightarrow 0$ the integrand converges to $f(y)g(y)$, and because f is integrable and $\|g * \varphi_\varepsilon\|_\infty \leq \|g\|_\infty < \infty$ the dominated convergence theorem implies that

$$\lim_{\varepsilon \rightarrow 0} \int f(y) (g * \varphi_\varepsilon)(y) dy = \int f(y) g(y) dy.$$

□

7.5 Fourier Transforms and Weak Convergence

Recall that a sequence $\{\mu_n\}_{n \geq 1}$ of Borel probability measures on \mathbb{R} converges weakly to a Borel probability measure μ (written $\mu_n \Rightarrow \mu$) if for every continuous function $u : \mathbb{R} \rightarrow \mathbb{R}$ with compact support,

$$\lim_{n \rightarrow \infty} \int u d\mu_n = \int u d\mu. \quad (7.20)$$

Recall also that a *sufficient* condition for this is that relation (7.20) holds for all C^∞ functions u with compact support. By the Parseval relation (7.16) (cf. Corollary 7.14),

$$\begin{aligned}\int u(x) d\mu(x) &= \frac{1}{2\pi} \int \hat{u}(\theta) \hat{\mu}(\theta) d\theta \quad \text{and} \\ \int u(x) d\mu_n(x) &= \frac{1}{2\pi} \int \hat{u}(\theta) \hat{\mu}_n(\theta) d\theta;\end{aligned}$$

consequently, since the Fourier transform \hat{u} decays rapidly at ∞ (cf. Homework), the convergence (7.20) will hold for any compactly supported C^∞ function u if

$$\lim_{n \rightarrow \infty} \hat{\mu}_n(\theta) = \hat{\mu}(\theta) \quad \text{for every } \theta \in \mathbb{R}. \quad (7.21)$$

This proves

Theorem 7.20. (*Lévy's Continuity Theorem*) A sequence μ_n of Borel probability measures on \mathbb{R} converge weakly to a Borel probability μ if and only if for every $\theta \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \hat{\mu}_n(\theta) = \hat{\mu}(\theta). \quad (7.22)$$

One can use Lévy's theorem as the basis for another proof of the central limit theorem.

Theorem 7.21. Let X_1, X_2, \dots be independent, identically distributed random variables with $EX_i = 0$ and $EX_i^2 = 1$, and write $S_n = \sum_{i=1}^n X_i$. Then the random variables S_n/\sqrt{n} converge in distribution to the standard normal distribution, that is, if μ_n is the distribution of S_n/\sqrt{n} then

$$\mu_n \implies \mu \quad \text{where} \quad \mu(B) = \frac{1}{\sqrt{2\pi}} \int_B e^{-x^2/2} dx. \quad (7.23)$$

Proof. It suffices to show that $Ee^{i\theta S_n/\sqrt{n}} \rightarrow e^{-\theta^2/2}$ for every θ . The random variables X_i have mean 0 and variance 1, so by Taylor's theorem their characteristic function $\varphi(\theta) := Ee^{i\theta X_n}$ satisfies

$$\varphi(\theta) = 1 - \frac{\theta^2}{2} \varphi''(\beta(\theta))$$

where $\beta(\theta)$ is a point of the real line intermediate between 0 and θ . Consequently, for any $\theta \in \mathbb{R}$,

$$\varphi(\theta/\sqrt{n}) = 1 - \frac{\theta^2}{2n} (1 + o(1)).$$

Therefore,

$$Ee^{i\theta S_n/\sqrt{n}} = \left(1 - \frac{\theta^2}{2n} (1 + o(1))\right)^n \rightarrow e^{-\theta^2/2}.$$

□

7.6 Lévy's Continuity Theorem

Lévy actually proved a much more useful extension of the Continuity Theorem 7.20 that can be used, in certain situations, to show that a given function is the Fourier transform of a Borel probability measure on \mathbb{R} .

Theorem 7.22. *The pointwise limit of a sequence of characteristic functions is a characteristic function if it is continuous at $\theta = 0$.*

Proof. This will be based on the Helly selection principle (Theorem 6.16). The strategy is as follows. Let $\varphi_n(\theta) = \hat{\mu}_n(\theta)$, where each μ_n is a Borel probability measure, and suppose that $\varphi_n(\theta) \rightarrow \varphi(\theta)$ for each $\theta \in \mathbb{R}$. If we knew that the sequence μ_n were tight, then Helly's theorem would imply that for any subsequence there is a weakly convergent subsequence – and in particular, the weak limit would be a Borel probability measure. But any such weak limit must have Fourier transform $\varphi(\theta)$; thus, by the Uniqueness Theorem, *there is only one possible weak limit μ* , and its Fourier transform must be $\varphi(\theta)$.

The gap in this argument is the tightness of the sequence μ_n . It is here that we will need the hypothesis that the limit function $\varphi(\theta)$ is continuous at $\theta = 0$. To make use of this hypothesis, we will find it convenient to utilize another characterization of tightness.

Lemma 7.23. *A sequence of probability measures μ_n on \mathbb{R} is tight if and only if for every $\varepsilon > 0$ there exists $\sigma^2 < \infty$ such that*

$$\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} d\mu_n(x) > 1 - \varepsilon. \quad (7.24)$$

Proof. Exercise. □

Corollary 7.24. *Let μ_n be a sequence of Borel probability measures on \mathbb{R} whose Fourier transforms $\hat{\mu}_n(\theta)$ converge pointwise to a function $\varphi(\theta)$ that is continuous at $\theta = 0$. Then the sequence μ_n is tight.*

Proof. Recall that the Plancherel-Parseval identity (7.16) holds for any bounded, continuous function f whose Fourier transform $\hat{f}(\theta)$ is integrable. This is the case for the function $f_\sigma(x) = \exp\{-x^2/2\sigma^2\}$: its Fourier transform is

$$\hat{f}_\sigma(\theta) = \sqrt{2\pi}\sigma \exp\{-\theta^2\sigma^2/2\},$$

which is certainly integrable. Thus, for any $\sigma > 0$ and each $n = 1, 2, \dots$,

$$\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} d\mu_n(x) = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\sigma^2\theta^2/2} \hat{\mu}_n(-\theta) d\theta = E\hat{\mu}_n(Z/\sigma),$$

where Z is a standard normal random variable.

By hypothesis, the functions $\hat{\mu}_n(\theta)$ converge pointwise to $\varphi(\theta)$, and the function $\varphi(\theta)$ is continuous at $\theta = 0$. Now since $\hat{\mu}_n(0) = 1$ for every n , it must also be the case that $\varphi(0) = 1$; and since $|\hat{\mu}_n(\theta)| \leq 1$ for every $\theta \in \mathbb{R}$, it must also be the case that $|\varphi(\theta)| \leq 1$. Hence, by the bounded convergence theorem,

$$\lim_{\sigma \rightarrow \infty} E\varphi(Z/\sigma) = \varphi(0) = 1,$$

and so for any $\varepsilon > 0$ there exists $\sigma < \infty$ such that

$$E\varphi(Z/\sigma) \geq 1 - \varepsilon.$$

But the pointwise convergence $\hat{\mu}_n(\theta) \rightarrow \varphi(\theta)$, together with another application of the bounded convergence theorem, implies that

$$\lim_{n \rightarrow \infty} E\hat{\mu}_n(Z/\sigma) = E\varphi(Z/\sigma) \geq 1 - \varepsilon.$$

Therefore, for all sufficiently large n ,

$$E\hat{\mu}_n(Z/\sigma) > 1 - 2\varepsilon,$$

and so by Lemma 7.23, the sequence μ_n is tight. □

□

7.7 The Symmetric Stable Laws

Theorem 7.25. For every real number $\alpha \in (0, 2)$ the function $\varphi_\alpha(\theta) = \exp\{-|\theta|^\alpha\}$ is a characteristic function.

Proof. Fix α , and let X_1, X_2, \dots be independent, identically distributed random variables with probability density $f_\alpha(x) = \frac{1}{2}\alpha|x|^{-\alpha-1}\mathbf{1}_{[1, \infty)}(|X|)$, and set $S_n = \sum_{i=1}^n X_i$. We will show that the characteristic functions of the random variables $S_n/n^{1/\alpha}$ converge pointwise to $\varphi_\alpha(\theta)$; Theorem 7.22 will then imply that φ_α is a characteristic function.

The characteristic function of X_i is the Fourier transform of the density f_α :

$$\varphi_{X_i}(\theta) = \hat{f}_\alpha(\theta) = \alpha \int_1^\infty \frac{\cos(|\theta|x|)}{x^{\alpha+1}} dx.$$

Consequently, as $|\theta| \rightarrow 0$,

$$1 - \varphi_{X_i}(\theta) = \alpha \int_1^\infty \frac{1 - \cos(|\theta|x|)}{x^{\alpha+1}} dx = \alpha|\theta|^\alpha \int_{|\theta|}^\infty \frac{1 - \cos y}{y^{1+\alpha}} dy \sim C|\theta|^\alpha,$$

where

$$C = \int_0^\infty \frac{1 - \cos y}{y^{1+\alpha}} dy;$$

the relation \sim follows from the dominated convergence theorem, using the fact that the (dominating) function $(1 - \cos y)y^{1+\alpha}$ is nonnegative and integrable on $(0, \infty)$. (This uses the hypothesis that $0 < \alpha < 2$.) Therefore, for any $\theta \in \mathbb{R}$ and any $n \geq 1$,

$$\varphi_{S_n/n^{1/\alpha}}(\theta) = \varphi_{X_1}(\theta/n^{1/\alpha})^n = \left(1 - \frac{C|\theta|^\alpha}{n}\right)^n,$$

and as $n \rightarrow \infty$ this converges to $\exp\{-|\theta|^\alpha\}$. \square

Exercise 7.26. Let $g(x)$ be any even probability density on \mathbb{R} such that $g(x) = f_\alpha(x)$ for all x outside some compact interval $[-A, A]$. Show that if Y_1, Y_2, \dots are independent, identically distributed with density g then

$$n^{-1/\alpha} \sum_{i=1}^n Y_i \implies \mu_\alpha$$

where μ_α is the symmetric stable law of exponent $\alpha \in (0, 2)$, that is, the unique probability measure with Fourier transform $\varphi_\alpha(\theta) = \exp\{-|\theta|^\alpha\}$. HINT: Show that the random variables Y_i can be constructed on a probability space that supports independent, identically distributed random variables X_i with density f_α in such a way that $X_i = Y_i$ on the event that $|X_i| > A$ or $|Y_i| > A$.