8 Laplace's Method and Local Limit Theorems

8.1 Fourier Analysis in Higher DImensions

Most of the theorems of Fourier analysis that we have proved have natural generalizations to higher dimensions, and these can be proved either by mimicking the one-dimensional arguments or by deducing them directly from the one-dimensional results with the help of the Fubini theorem. Here we will look only at random vectors that take values in the integer lattice $\mathbb{Z}^k$, with the aim of developing an inversion formula that will allow us to settle the question of recurrence/transience for $k-$dimensional random walk.

Definition 8.1. The *characteristic function* of a $k-$dimensional random vector X is the function $\varphi_X : \mathbb{R}^d \to \mathbb{C}$ defined by

$$\varphi_X(\theta) = Ee^{i\langle \theta, X\rangle}$$

where $\langle u, v\rangle$ denotes the inner product of the vectors u, v. If the random vector X takes values in $\mathbb{Z}^k$ then the characteristic function $\varphi_X(\theta)$ is $2\pi-$periodic in each coordinate θ_j.

Lemma 8.2. (Orthogonality Relations) *The exponential functions $e^{i\langle m,\theta\rangle}$ are orthonormal in $L^2[-\pi,\pi]^k$, that is, for any two elements $m, n \in \mathbb{Z}^k$,*

$$(2\pi)^{-k} \iint \cdots \int_{[-\pi,\pi]^k} e^{i\langle m,\theta\rangle} e^{-i\langle n,\theta\rangle}\, d\theta_1 d\theta_2 \cdots d\theta_k = \delta_{m,n}. \tag{8.1}$$

Corollary 8.3. *If X takes values in $\mathbb{Z}^k$ then*

$$P(X = m) = (2\pi)^{-k} \iint \cdots \int_{[-\pi,\pi]^k} \varphi_X(\theta) e^{-i\langle m,\theta\rangle}\, d\theta_1 d\theta_2 \cdots d\theta_k. \tag{8.2}$$

For simple random walk S_n on $\mathbb{Z}^k$, the step distribution has characteristic function $\varphi(\theta) = \frac{1}{k}\sum_{j=1}^k \cos\theta_j$; therefore, the return probabilities are given by

$$P(S_n = 0) = (2\pi)^{-k} \iint \cdots \int_{[-\pi,\pi]^k} \frac{1}{k}\sum_{j=1}^{k} \cos\theta_j \, d\theta_1 d\theta_2 \cdots d\theta_k. \tag{8.3}$$

8.2 Laplace's Method

The first case of the central limit theorem, for sumes of independent, identically distributed Bernoulli random variables, was proved in the 1730s by De Moivre. De Moivre proof relied on Stirling's formula $n! \sim n^n e^{-n}\sqrt{2\pi n}$ and the fact that the binomial distribution can be explicitly written in terms of factorials. Some 40 (?) years later Laplace provided a new – and better – approach, using what is now called the *Laplace method* of asymptotic expansion. This is a fundamentally important technique in mathematical analysis, and one which is especially useful in probability and statistics, and it is the easiest route to the *local central limit theorem.* It relies on the following fact, which I will take as known.

Lemma 8.4.
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = 1.$$

The Laplace method is a technique for obtaining sharp approximations to integrals of the form (or similar in form to)

$$J(n) := \int_a^b e^{ng(x)}\,dx \tag{8.4}$$

where g is a smooth real-valued function that attains its max in $[a,b]$ *uniquely* at an interior point, which I will assume (without loss of generality) is the origin. Since g is maximal at $x=0$, its first derivative is $g'(0)=0$ and its second derivative is non-positive. There is no loss of generality in assuming that $g(0)=0$ (because subtracting a constant a from g just multiplies the entire integral by e^{-an}). Laplace's great insight was that if in addition the second derivative $g''(0)<0$ is nonzero, then when n is large the integrand $e^{ng(x)}$ spikes very sharply at 0, and in a neighborhood of 0 of width $3/\sqrt{n}$ or $5/\sqrt{n}$ or so the two-term Taylor series approximation to g gives

$$e^{ng(x)} \approx e^{ng''(0)x^2/2},$$

so the spike looks very much like a Gaussian distribution. Formally replacing the integrand by this Gaussian distribution and integrating (using Lemma 8.4 together with the substitution $y=\sqrt{n}x$) then yields the approximation

$$J(n) := \int_a^b e^{ng(x)}\,dx \approx \int_{-\infty}^{\infty} e^{ng''(0)x^2/2}\,dx = \frac{\sqrt{2\pi}}{\sqrt{n(-g''(0))}}.$$

Theorem 8.5. *(Laplace) Assume that g is a smooth (at least C^2) function on the interval $[a,b]$, where $a<0<b$, and assume that*

(a) $g(0)=0$*;*
(b) $g''(0)<0$*; and*
(c) $g(x)<0$ *for every $x\in[a,b]$ except $x=0$.*

Then as $n\to\infty$,
$$J(n) := \int_a^b e^{ng(x)}\,dx \sim \frac{\sqrt{2\pi}}{\sqrt{-ng''(0)}}. \tag{8.5}$$

The symbol $\sim$ means that the limit of the *ratio* of the two sides converges to 1 as $n\to\infty$ (equivalently, the *relative* error in the approximation goes to 0).

Proof of Theorem 8.5. The integral J_n will be analyzed by breaking the range of integration into two parts: (i) a small interval $[-\delta,\delta]$ containing 0, and (ii) everything else, that is, $[a,-\delta]\cup[\delta,b]$. The second region (ii) is easier, so we begin with this. By hypothesis, the function g is strictly less than 0 at every $x\neq 0$, and it is continuous on the intervals $[a,-\delta]$ and $[\delta,b]$, so there exists $\gamma=\gamma(\delta)>0$ such that $g(x)\leq -\gamma$ for every $x\in[a,-\delta]\cup[\delta,b]$. Consequently,

$$\int_a^{-\delta} e^{ng(x)}\,dx+\int_\delta^b {}^{ng(x)}\,dx\leq e^{-n\gamma}(b-a).$$

Since $e^{-\gamma n}$ is negligible compared to $n^{-1/2}$ as $n\to\infty$, to complete the proof it will suffice to show that for any $\varepsilon>0$ there is a $\delta>0$ such that for all large n,

$$1-\varepsilon<\sqrt{n}\int_{-\delta}^{\delta} e^{ng(x)}\,dx\Big/\sqrt{2\pi/(-g''(0))}<1+\varepsilon.$$

Now for any $\varepsilon>0$ there exists $\delta>0$ such that in the interval $[-\delta,\delta]$ the two-term Taylor series approximation to $g(x)$ is accurate to within a factor $(1\pm\varepsilon)^2$, that is, for all $x\in[-\delta,\delta]$,

$$(1-\varepsilon)^2 g''(0)x^2/2<g(x)<(1+\varepsilon)^2 g''(0)x^2/2.$$

Hence, for $x\in[-\delta,\delta]$ the integrand $e^{ng(x)}$ is bounded above and below by Gaussian densities with variances $((1\pm\varepsilon)^2/(-g''(0)n)$. Consequently, the integral $\int_{-\delta}^{\delta} e^{ng(x)}\,dx$ is bounded above and below by the integrals of the bounding Gaussian densities, which by Lemma 8.4 are

$$(1\pm\varepsilon)\sqrt{\frac{2\pi}{-g''(0)n}}.$$

(Note: because the Gaussian densities have variances of order $1/n$, the total mass outside $[-\delta,\delta]$ is exponentially small, by Homework 6, problem 3.)

$\square$

We have already encountered a class of integrals that very similar in form to (8.4), in the Fourier inversion formula (**??**). In general, the characteristic functions $\varphi_X(\theta)$ of integer-valued random variables X are not real-valued, and furthermore the integrals (**??**) contain an additional factor $e^{-im\theta}$, so 8.5 does not apply directly. But there are some important special cases where Laplace's theorem does apply directly.

Example 8.6. Let $X_1,X_2,\dots$ be independent, identically distributed Rademacher random variables with partial sums S_n. As we have seen, the characteristic function of S_n is

$$\varphi_{S_n}(\theta)=\cos^n\theta.$$

Clearly, when n is odd the return probability $P\{S_n=0\}=0$. When n is even, the characteristic function $\cos^n\theta$ has period π, and so the Fourier inversion formula (**??**) implies that

$$P\{S_n = 0\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^n\theta\,d\theta$$
$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos^n\theta\,d\theta$$

Hence, by Laplace's theorem,

$$P\{S_{2n} = 0\} \sim \sqrt{\frac{1}{\pi n}}.$$

Example 8.7. Let $\{p_n\}_{n\in\mathbb{Z}}$ be a *symmetric* probability distribution on the integers (that is, $p_n = p_{-n}$) with variance $0 < \sigma^2 < \infty$, and let $X_1, X_2, \ldots$ be independent, identically distributed with distribution

$$P\{X_n = m\} = p_m.$$

Then the characteristic function $\varphi(\theta) = Ee^{i\theta X_1}$ is real-valued and has even, and has second derivative $\varphi''(0) = -\sigma^2$. *Assume* that the support of the distribution $\{p_n\}_{n\in\mathbb{Z}}$ is not contained in an arithmetic progression $am + b$ where $a \geq 2$. Then by Proposition **??**, $|\varphi(\theta)| < 1$ for every $\theta \in [-\pi,\pi]$ except $\theta = 0$. By the Fourier inversion formula (**??**),

$$P\{S_n = 0\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(\theta)^n\,d\theta.$$

Since φ is real-valued and has unique max at $\theta = 0$, Laplace's theorem applies, so we conclude that

$$P\{S_n = 0\} \sim \frac{1}{\sqrt{2\pi n}\sigma}.$$

Exercise 8.8. (a) Show that $n! = \int_0^\infty x^n e^{-x}\,dx$.
(b) Deduce that $n! = n^{n+1}e^{-n}\int_0^\infty y^n e^{-(y-1)n}\,dy$.
(c) Adapt the proof of Laplace to show that $n! \sim n^n e^{-n}\sqrt{2\pi n}$.

Laplace's method extends without great difficulty to multiple integrals. The only differences with the one-dimensional case are (i) one must replace the use of Taylor's approximation to $g(x)$ near the origin by a multivariate Taylor approximation; and (ii) the Gaussian densities that bound $e^{ng(x)}$ for x near the origin must be replaced by multivariate Gaussian densities. Following is a particular case of interest, where the relevant multivariate Gaussian density is just the product of one-dimensional Gaussian densities.

Theorem 8.9. *Let $g : [-a,a]^d \to \mathbb{R}$ be a smooth function that satisfies the following hypotheses:*

(a) $g(0) = 0$*;*
(b) $g(x) < 0$ *for all* $x \neq 0$*;*
(c) $D^2 g(0) = -\sigma^2 I$ *for some* $0 < \sigma^2 < \infty$*,*

where $D^2 g$ is the $d \times d$ matrix of second partial derivatives and I is the $d \times d$ identity matrix. Then as $n \to \infty$,

$$\int_{[-a,a]^d} e^{ng(x)}\,dx \sim \sigma^{-d} \left(\frac{2\pi}{n}\right)^{d/2}. \tag{8.6}$$

Example 8.10. Let S_n be the location of a simple nearest neighbor random walk on the $d-$dimensional integer lattice after n steps. Then

$$P\{S_{2n} = 0\} \sim \left(\frac{d}{\pi n}\right)^{d/2}$$

Consequently, the expected number of returns $\sum_{n=1}^{\infty} P\{S_{2n} = 0\}$ to the origin is finite if $d \geq 3$, Thus simple random walk in 3 dimensions and higher is *transient*.