STATISTICS 312: STOCHASTIC PROCESSES II
HOMEWORK ASSIGNMENT 9
DUE THURSDAY MAY 31

In the following problems , $W(t)$ is a either a standard Wiener process or a Wiener process started at some $x \in \mathbb{R}$, depending on the context. The random variables $M(t)$ and $M^-(t)$ are the maximum and minimum up to time t, and $\tau(a)$ is the first time that $W(t)$ visits a; thus,

$$\begin{aligned} M(t) &= \max \{W(s) \,:\, s \leq t\}, \\ M^-(t) &= \min \{W(s) \,:\, s \leq t\} \qquad \text{and} \\ \tau(a) &= \min \{t : W(t) = a\}. \end{aligned}$$

1. **Reflection Principle.**

(A) Show that for any $t > 0$ the random variable $|W(t)|$ has the same distribution as the random variable $M(t) - W(t)$. HINT: Begin by calculating

$$P\{M(t) \geq a \text{ and } W(t) \leq a - b\}$$

for $a, b > 0$. HINT: Reflection principle.

(B) Calculate the transition probability densities for the process $|W_t|$. That is, for each $t > 0$ and each $x \geq 0$ find a probability density $p_t^+(x, y)$ in $y \geq 0$ such that for every interval $[a, b] \subseteq [0, \infty)$,

$$P(|W_t| \in [a, b] \,|\, |W(0) = x) = \int_a^b p_t^+(x, y)\, dy$$

HINT: Reflection principle.

(C) Calculate the transition probability densities $q_t^+(x, y)$ for the process $Y_t = M_t - W_t$. NOTE: To start the process Y_t in the initial state $Y_0 = x$, do the following: Let W_t be a standard Wiener process started at 0, and redefine

$$\begin{aligned} M_t &= x \quad \text{if } t \leq \tau(x); \\ M_t &= \max \{W(s) \,:\, s \leq t\} \quad \text{if } t \geq \tau(x). \end{aligned}$$

There should be an interesting connection between the answers to (B) and (C). HINT: Reflection principle.

2. **Brownian motion with absorption at** 0**.** Define Brownian motion with absorption at 0 by $Y_t = W_{t \wedge \tau(0)}$, that is, Y_t is the process that follows the Brownian path until the first visit to 0, then sticks at 0 forever after. Calculate the transition probability densities $p_t^0(x, y)$ of Y_t.

3. **Skorohod Embedding.** Let $a < 0 < b < c$, and let p_a, p_b, p_c be a probability distribution on a, b, c with mean zero, that is,

$$ap_a + bp_b + cp_c = 0$$

Show that there is a stopping time τ for Brownian motion such that

$$\begin{aligned} P(W(\tau) = a) &= p_a, \\ P(W(\tau) = b) &= p_b, \\ P(W(\tau) = c) &= p_c. \end{aligned}$$

HINT: Let $d = (bp_b + cp_c)/(p_b + p_c)$ be the conditional mean given that the outcome is positive. Show that τ can be constructed by *first* running the Brownian motion until the first time it hits either a or d, and *then*, if it hits d before a, let it continue until it hits either b or c.

NOTE: If you are so inclined, you could now give a proof by induction that for every discrete probability distribution $\mathcal{Q}$ with mean zero there is a stopping time τ such that $W(\tau)$ has distribution $\mathcal{Q}$.

4. **Reflection Principle for Parallel Mirrors.** Show that the joint distribution of $M(t)$ and $M^-(t)$ is given by

$$\begin{aligned} P\left\{M(t) > b \text{ and } M^-(t) < -a\right\} &= \sum_{k=0}^{\infty} 2P\left\{W(t) > b + (2k+1)(b+a)\right\} \\ &+ \sum_{k=0}^{\infty} 2P\left\{W(t) > a + (2k+1)(b+a)\right\} \\ &- \sum_{k=0}^{\infty} 2P\left\{W(t) > b + 2k(b+a)\right\} \\ &- \sum_{k=0}^{\infty} 2P\left\{W(t) > a + 2k(b+a)\right\} \end{aligned}$$

or something like this. HINT: The key ingredients are the reflection principle and the inclusion/exclusion formula. Here's a start: In order that $M(t) > b$ and $M^-(t) < -a$, one of two things must occur: (i) visit b, then visit $-a$, or (ii) visit $-a$, then visit b. Evaluate the probabilitis of these events using the reflection principle and then add them. Unfortunately, you will have overcounted the intersection, so you will now have to subtract its probability. How can the intersection of (i) and (ii) occur? One of two things must happen: (iii) visit b, then visit $-a$, then revisit b, or (iv) visit $-a$, then visit b, then revisit $-a$. Now you have overcounted again, so you will have to go back and add the probability of the intersection of (iii) and (iv). And so on.

5*. **Local Maxima of the Brownian Path.** A continuous function $f(t)$ is said to have a *local maximum* at $t = s$ if there exists $\varepsilon > 0$ such that

$$f(t) \le f(s) \qquad \text{for all } t \in (s - \varepsilon, s + \varepsilon).$$

(A) Prove that if the Brownian path $W(t)$ has a local maximum w at some time $s > 0$ then, with probability one, it cannot have a local maximum at some later time s^* with the same value w. HINT: Use the Strong Markov Property and the fact that the rational numbers are countable and dense in $[0, \infty)$.

(B) Prove that, with probability one, the times of local maxima of the Brownian path $W(t)$ are dense in $[0, \infty)$

(C) Show that the *values* $W(t)$ at the times of local maxima are the same as the values $a > 0$ where the first-passage time process $\tau(a)$ is discontinuous in a.

(D) Conclude that, with probability one, the set of local maxima of the Brownian path $W(t)$ is *countable.*