STATISTICS 313: STOCHASTIC PROCESSES II
HOMEWORK ASSIGNMENT 7
DUE TUESDAY JUNE 11

In the following problems , $W(t)$ is a standard Wiener process, $M(t)$ is the maximum up to time t, and for any $a > 0$, $\tau(a)$ is the first time that $W(t)$ visits a; thus,

$$M(t) = \max\{W(s) : s \leq t\},$$
$$\tau(a) = \min\{t : W(t) = a\}.$$

Problem 1. Show that for any $t > 0$ the random variable $M(t) - W(t)$ has the same distribution as the random variable $|W(t)|$. HINT: Begin by calculating

$$P\{M(t) \geq a \text{ and } W(t) \leq a - b\}$$

for $a, b > 0$. You may find the reflection principle helpful.

NOTE: In fact even more is true (as was discovered by P. Lévy): for any times $0 < t_1 < t_2 < \cdots < t_m$ the *joint* distribution of the random variables $\{M(t_i) - W(t_i) : 1 \leq i \leq m\}$ is the same as the joint distribution of $\{|W(t_i)| : 1 \leq i \leq m\}$. If you are feeling ambitious you might try to prove this by induction on m.

Problem 2. *Trigonometry.* Let $T_- = \max\{t < 1 : W_t = 0\}$ and $T_+ = \min\{t > 1 : W_t = 0\}$.

(a) Show that with probability one, $0 < T_- < 1 < T_+$.

(b) Prove that T_- is *not* a stopping time, but that T_+ is.

(c) Show that $P\{T_+ \leq s\} = (2/\pi) \arccos(1/\sqrt{s})$ for all $s \geq 1$.

(d) Show that $P\{T_+ \geq t \text{ and } T_- \leq s\} = (2/\pi) \arcsin \sqrt{s/t}$ for all $0 < s < 1 < t < \infty$.

(e) What is the density of T_-/T_+?

Problem 3. *Killed Brownian Motion.* For fixed constants $a < 0 < b$ define $T = T_{a,b}$ to be the first time that the Wiener process hits either a or b. This time is finite, because $T_{a,b} \leq \tau(b)$. For $a < x, y < b$ and $t > 0$ define

$$p^*_t(x, y) = P^x\{W_t \in dy \text{ and } T > t\}. \tag{1}$$

(Here P^x is a probability measure under which W_t is a Brownian motion started at x.) This is a *sub*-probability density which is bounded above by the Gaussian density with mean 0 and variance t.

(a) Consider the special case where $a = -1/2$ and $b = 1/2$. Use the Wald identity

$$E^x \exp\{i\theta W_\tau + \theta^2 \tau/2\} = e^{i\theta x}. \tag{2}$$

to show that for each *odd* integer k,

$$\int_{-\frac{1}{2}<y<\frac{1}{2}} p^*_t(x, y) \cos(k\pi y)\, dy = \exp\{-(\pi k)^2 t/2\} \cos(k\pi x).$$

(b) Conclude from (a) that $p^*_t(y)$ has the Fourier series expansion

$$p^*_t(y) = \sum_{k \text{ odd}} e^{-(\pi k)^2 t/2} 2\cos(\pi k y).$$

HINT: Use the fact that the sequence of functions $\{2^{-1/2}e^{\pi ikx}\}_{k\in\mathbb{Z}}$ is an orthonormal basis of $L^2[-1,1]$ to show that the sequence $\{\cos(k\pi x)\}_{k\text{odd}}$ is an orthonormal basis of $L^2[-\frac{1}{2},\frac{1}{2}]$.

(c) Show that $\lim_{y\to\pm 1/2} p_t(x,y)=0$. HINT: Show that $p_t(y)\le P\{W_t\in dy \text{ and } \tau_{1/2}>t\}$ where $\tau_{1/2}$ is the time of first passage to 1/2. You should have an exact formula for the latter density from the reflection principle.

Problem 4. *Reflection Principle for Parallel Mirrors.* It is also possible to write an infinite series expansion for the density $p_t^*(x,y)$ defined in (1) using a revved-up form of the Reflection Principle. Assume for simplicity that $a=-1/2$, $b=1/2$, and $x=0$. Let W_t be the standard Wiener process, and let $p_t(y)$ be the density at time t (the Normal density with mean zero and variance t). Use the Reflection Principle (repeatedly) together with the law of inclusion-exclusion to show that

$$p_t^*(0,y)=\sum_{k=-\infty}^{\infty}(-1)^k p_t((-1)^k y+k) \tag{3}$$

$$=\sum_{k=-\infty}^{\infty}(-1)^k \exp\{-((-1)^k y+k)^2/2t\}/\sqrt{2\pi t}.$$

NOTE: The fact that the two infinite series in this problem and in problem 3 (b) define the same function $p_t^*(y)$ also follows from the *Poisson summation formula* for Fourier series. You can find this formula in Feller, vol. 2. However, you should not use the Poisson summation formula to solve this problem.

Problem 5. * *Local Maxima of the Brownian Path.* A continuous function $f(t)$ is said to have a *local maximum* at $t=s$ if there exists $\varepsilon>0$ such that

$$f(t)\le f(s)\qquad \text{for all } t\in(s-\varepsilon,s+\varepsilon).$$

(a) Prove that for any fixed time $t0$,

$$P\{W_t=M_t\}=0.$$

HINT: Time reversal.

(b) For any two rational times $q_0<q_1$ let $M(q_0,q_1)=\max_{t\in[q_0,q_1]}W_t$ be the maximum of the Brownian path in the time interval $[q_0,q_1]$. Prove that with probability one,

$$W_{q_0}<M(q_0,q_1)\quad\text{and}\quad W_{q_1}<M(q_0,q_1)$$

for *all* pairs of rationals q_0,q_1.

(c) Conclude that with probability one, the set of times at which the Brownian path $W(t)$ has a local maximum is dense in $[0,\infty)$.

(d) Prove that with probability one, for any two pairs of rational times $q_0<q_1$ and $q_2<q_3$,

$$M(q_0,q_1)\ne M(q_2,q_3).$$

HINT: Strong Markov property.

(e) Prove that, with probability one, the set of local maxima of the Brownian path $W(t)$ is *countable*.