STATISTICS 313: STOCHASTIC PROCESSES II
HOMEWORK ASSIGNMENT 23
DUE THURSDAY MAY 30

Problem 1. Random Walk on a Complete Graph. The *complete graph* on n vertices, denoted by K_n, is the graph with vertex set $[n] = \{1, 2, 3, \ldots, n\}$ such that for every pair of vertices i, j there is an edge $e(\{ij\})$ with endpoints $i$and j. Consider the electrical network on K_n in which every edge is assigned resistance 1. Calculate the effective resistance between vertices 1 and n.

Problem 2. Ehrenfest Random Walk. Calculate the escape probability $p_{\text{escape}}(\mathbf{0}; \mathbf{1})$ for the Ehrenfest random walk on $\mathbb{Z}_2^N$, where $\mathbf{0}$ and $\mathbf{1}$ are the extreme corners of the hypercube, that is,

$$\mathbf{0} = (0, 0, \ldots, 0) \quad \text{and} \quad \mathbf{1} = (1, 1, \ldots, 1).$$

HINT: Use the series, parallel, and shorting laws discussed in the notes.

Problem 3. Shorting and Effective Resistance. Let $\{C_e\}_{e \in \mathcal{E}}$ be a positive conductance function on a connected graph (or multi-graph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Let $s, t \in \mathcal{V}$ be two distinguished vertices, and let $x, y \in \mathcal{V} - \{s, t\}$ be two vertices other than s, t. Consider the electrical network obtained by removing the vertices x, y and replacing them by a single vertex z, and then re-routing all the conducting edges in the original graph that went through x or y to z. Show that the effective resistance between s and t in the new network is no larger than the effective resistance between s and t in the original network.

Problem 4. Bead Game. This problem concerns the following stochastic process on a finite graph G with conductance function C_{xy}. At each vertex of $\mathcal{G}$ there is a bead, which is either black or white. At each (discrete) time $n = 0, 1, \ldots$ an edge of the graph is chosen at random with probability C_{xy}/C, where C is the sum of the conductances in the network, and the beads at x and y are interchanged. When a bead reaches a it is immediately painted black, and when it reaches b it is immediately painted white.

(A) Let $h(x)$ be the voltage at x when a unit voltage is imposed between a and b (i.e., a 1 volt battery is hooked up to a and b). Show that regardless of the initial configuration of black and white beads on the vertex set $\mathcal{V}$ the system reaches a statistical equilibrium in which $h(x)$ is the probability of a black bead occupies vertex x.

(B) Let I be the electrical current flow in the network when a unit voltage is imposed between a and b. Show that there is a constant $\gamma > 0$ such that for every edge xy, the net flow of black beads across the edge xy is γI_{xy}.

Problem 5. More Electrical Connections. Let X_n be an irreducible, reversible Markov chain on a finite state space $\mathcal{X}$ with transition probability matrix $\mathbb{P} = (p_{x,y})$ and stationary distribution π_x. Define $C_{x,y} = \pi_x p_{x,y}$ to be the conductance function of the corresponding electrical network. Let s, t be two distinct vertices of $\mathcal{X}$, and set

$$T = \min\{n \geq 1 \,:\, X_n = t\} \quad \text{and} \quad \tau = \min\{n > T \,:\, X_n = s\}.$$

(A) Define

$$G(s,y) = E^s \sum_{n=0}^{T-1} \mathbf{1}\{X_n = y\}$$

to be the expected number of visits to y before first hitting t. Show that

$$G(s,y) = \pi(y)v(y)R_{\text{eff}}(s;t)$$

where $R_{\text{eff}}(s;t)$ is the effective resistance between s and t and $v(x)$ is the voltage at x when the voltages at s and t are held at $v(s) = 1$ and $v(t) = 0$. HINT: Let $f(y) = G(s,y)/\pi(y)$. What kind of function is $f(y)$?

(B) For any two (distinct) neighboring vertices x, y, let $S_{x,y}$ be the number of times that the Markov chain jumps from x to y before reaching the terminus t, that is,

$$S_{x,y} = \sum_{n=0}^{T-1} \mathbf{1}\{X_n = x \text{ and } X_{n+1} = y\}.$$

Show that

$$E^a S_{x,y} - E^a S_{y,x} = J_{x,y}$$

where J is the electrical current flow in the network when the voltages at s and t are held at

$$V(s) = R_{\text{eff}}(s;t) \quad \text{and} \quad V(t) = 0.$$

(C) *Loop-Erased Random Walk.* Consider the path $X = \{X_n\}_{0 \le n \le T}$ of the Markov chain up until the time T when it first hits the terminal vertex t. In general, this path will make *loops*, that is, it will occasionally revisit a vertex x that it had visited earlier. Define L to be the path obtained from X by erasing all loops as they form. For instance, if

$$X = sabcadeqegkt \quad \text{then} \quad L = sadegkt.$$

(i) Show that L visits any vertex x at most once, and consequently, crosses any edge at most once.
(ii) Let $\tilde{S}_{x,y}$ be the number of times that the loop-erased path L crosses from x to y. Show that $E\tilde{S}_{x,y} = ES_{x,y}$.

Problem 6. Spanning Trees. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a finite, connected graph. A *spanning tree* of $\mathcal{G}$ is a connected subgraph $\mathcal{T} = (\mathcal{V}, \mathcal{E}^*)$ with no cycles. (Note: The vertex set must be the same as that of $\mathcal{G}$, and the edge set $\mathcal{E}^*$ must be a subset of the edge set $\mathcal{E}$ of $\mathcal{G}$.) Spanning trees are gotten by removing edges from cycles one at a time until no cycles remain. There are, in general, many spanning trees of a connected graph $\mathcal{G}$.

(A) Show that if T is a spanning tree then for any two *distinct* vertices $x, y \in \mathcal{V}$ there is a *unique* self-avoiding path in T from x to y.

(B) Assume now that there are two distinct vertices $s, t \in \mathcal{V}$, the *source* and the *terminus*. For any two vertices x, y that share an edge in $\mathcal{G}$, define $N(s,x,y,t)$ to be the number of spanning trees T of $\mathcal{G}$ in which the unique path from s to t crosses the edge from x to y (that is, the path must be of the form $s, x_1, \dots, x_j, x, y, \dots, x_m, t$). Define

$$J(x,y) = N(s,x,y,t) - N(s,y,x,t).$$

Prove that J is a flow with source s and sink t. What is the size $J(s+)$ of the flow? HINT: Write J as a sum $J = \sum_T J^T$ where the sum is over all spanning trees T and $J^T(x,y) = +1, -1$, or 0

depending on whether the unique path in T from s to t crosses xy, or crosses yx, or doesn't cross in either direction. Show that each each J^T is a flow with source s and sink t.

(C) Prove that the flow J is acyclic, and conclude that it must be a gradient flow, that is, there is a function $w : \mathcal{V} \to \mathbb{Z}$ such that $J(x, y) = w(x) - w(y)$. HINT: Use the fact that a spanning tree has no cycles.

(D) Let N be the total number of spanning trees of $\mathcal{G}$, and let w be the function obtained in (C). Define

$$u(x) = w(x)/N.$$

Show that u is the voltage function of the electrical network in which each edge of $\mathcal{G}$ has conductance 1 and the total current through s is 1.

Problem 7. Helly's Selection Theorem. Let $\{\mu_n\}_{n\geq 1}$ be a sequence of (Borel) probability measures on the unit interval $[0, 1]$. *Helly's selection theorem* states that there is a subsequence $\{\mu_{n_k}\}_{k\geq 1}$ that converges in distribution to some (Borel) probability measure μ on $[0, 1]$.

(A) Show that there is a subsequence $\{\mu_{n_k}\}_{k\geq 1}$ such that for every $m \geq 0$

$$\lim_{k\to\infty} \int_{[0,1]} x^m \, d\mu_{n_k}(x) := \alpha_m$$

exists. HINT: Bolzano-Weierstrass plus Cantor's diagonal argument.

(B) Conclude that for every continuous function $f : [0, 1] \to \mathbb{R}$,

$$\lim_{k\to\infty} \int_{[0,1]} f(x) \, d\mu_{n_k}(x) := \Lambda(f)$$

exists. HINT: Weierstrass Approximation Theorem.

It requires a bit more machinery (e.g., either the Riesz Representation Theorem or the Caratheodory Extension Theorem) to prove rigorously that there is a Borel probability measure λ on $[0, 1]$ such that $\Lambda(f) = \int f \, d\lambda$ for every continuous function f. However, given the result of (B) it is not difficult to understand why this should be the case: for any interval $[0, x]$ the indicator function $\mathbf{1}_{[0,x]}$ can be very closely approximated by continuous functions, and so (B) shows how to figure out how much mass λ should assign to $[0, x]$. In other words, the convergence in (B) determines the c.d.f. of λ. The technical step (Caratheodory Extension Theorem) is to show that for any c.d.f. there is a matching probability measure λ.

Problem 8. Exchangeable Sequences. A sequence $X_1, X_2, \ldots$ of random variables is said to be *exchangeable* if for every integer $m \geq 1$ and every permutation σ of $[m]$,

$$(X_1, X_2, \ldots, X_m) \stackrel{\mathcal{D}}{=} (X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(m)}).$$

Assume that $X_1, X_2, \ldots$ is an exchangeable sequence of *Bernoulli* random variables. Let $S_n = \sum_{j=1}^n X_j$ and $R_n = n - S_n$.

(A) Prove that the sequence $(R_n, S_n)_{n\geq 0}$ is a Markov chain on $\mathbb{Z}_+^2$.

(B) Say that a path in the integer lattice is *admissible* if every step is either up one (i.e., add (0,1)) or one to the right (i.e., add $(1, 0)$). Show that for any admissible path γ from $(0, 0)$ to (r, s), the probability that the first $r + s$ steps of the Markov chain follow the path γ is

$$\frac{P\{S_n = s\}}{N(r, s)}$$

where $N(r,s)$ is the number of admissible paths from $(0,0)$ to (r,s).

(C) Use the result of part (B) to write a simple expression for

$$P(X_1 = e_1, X_2 = e_2, \ldots, X_k = e_k \,|\, S_n = m)$$

valid for any $m \geq k$ and any sequence e_i of 0s and 1s.

(D) Let μ_n be the distribution of S_n/n, that is, the probability measure on $[0,1]$ that puts mass $P\{S_n = k\}$ at the point k/n for every integer $k = 0, 1, \ldots, n$. By the Helly Selection Theorem there is a subsequence μ_{n_k} that converges in distribution to a probability measure μ on $[0,1]$. Use the result of (C) to show that

$$P\{X_1 = e_1, X_2 = e_2, \ldots, X_k = e_k\} = \int_{[0,1]} \theta^{\sum_{j=1}^k e_j} (1-\theta)^{k - \sum_{j=1}^k e_j} \, d\mu(\theta).$$

The existence of such a representation is known as *de Finetti's Theorem*. What it asserts is that *every exchangeable sequence of Bernoulli random variables is a mixture of i.i.d. Bernoulli processes*. The probability measure μ is the *mixing measure*. The de Finetti representation can be interpreted as a two step algorithm for simulating the sequence $X_1, X_2, \ldots$: first, draw Θ at random with distribution μ; then, given $\Theta = \theta$, generate i.i.d. Bernoulli θ random variables.