STATISTICS 312: STOCHASTIC PROCESSES II
HOMEWORK ASSIGNMENT 5
DUE TUESDAY MAY 21

1. **Simple random walk on $\mathbb{Z}_m$.** Fix $m \geq 1$, and consider the lazy simple random walk S_n on the discrete circle $\mathbb{Z}_m$ started at the group identity 0, that is, the Markov chain whose transition probabilities are

$$\begin{aligned} p(x,x) &= 1/2 \\ p(x,x+1) &= 1/4 \\ p(x,x-1) &= 1/4 \end{aligned}$$

where addition is mod m. As we know by now, the stationary distribution for this Markov chain is the uniform distribution π on $\mathbb{Z}_m$. Show that when m is large the mixing time of this Markov chain is "about m^2". More precisely, prove the following statements.

(a) For any $\varepsilon > 0$ there exists $\delta > 0$ such that if $n \leq \delta m^2$ then

$$\|\mathcal{D}(S_n) - \pi\|_{TV} \geq 1 - \varepsilon.$$

(b) For any $\varepsilon > 0$ there exists $C < \infty$ such that if $n \geq Cm^2$ then

$$\|\mathcal{D}(S_n) - \pi\|_{TV} \leq \varepsilon.$$

HINT: CLT.

Do the following problems in Lecture Notes B: Convergence Rates of Markov chains.

2. Problem 6 (Poissonization).
3. Problem 7 (Coin Collector).
4. Problem 8 (Top-to-Random Shuffling).
5. Problem 9 (Mixing Rate for Random-to-Top Shuffling).

6.* **Modified Ehrenfest random walk.** Consider the following variation of the Ehrenfest random walk: at each step, a ball is chosen at random and then moved to the opposite urn with probability p. (Note: The Ehrenfest random walk discussed in class and in the notes is the special case $p = 1/2$.)

(a) Check that the stationary distribution is the uniform distribution π.

(b) Verify that $S_n = S_0 + \sum_{j=1}^{n} \xi_j$ where $\xi_1, \xi_2, \ldots$ are independent, identically distributed (and addition is done mod 2).

(c) Find the eigenvalues and eigenfunctions of the transition probability operator.

(d)* What is the mixing time? More precisely, show that there is a positive constant $C = C_p$ such that for every $\varepsilon > 0$

(i) If $n = (1-\varepsilon)CN \log N$ then $\lim_{N\to\infty} \|\mathcal{D}(S_n) - \pi\|_{TV} = 1$, and

(ii) If $n = (1+\varepsilon)CN \log N$ then $\lim_{N\to\infty} \|\mathcal{D}(S_n) - \pi\|_{TV} = 0$.