

STATISTICS 312: STOCHASTIC PROCESSES II
HOMEWORK ASSIGNMENT 4
DUE TUESDAY MAY 14

1. **Distributions with Long Tails:** The purpose of this exercise is to introduce you to the strange world of distributions with infinite first moment. You will see that the *tail behavior* is mirrored by the behavior of the characteristic function near $\theta = 0$. You will also learn how to modify Laplace's method for use in situations where the integrand is not differentiable at the point where it attains its max.

(A) First an instructive example. For any $\alpha > 1$ define the $\zeta(\alpha)$ -distribution to be the symmetric probability distribution on the integers with probability mass function

$$f_\alpha(k) = \begin{cases} 1/(2\zeta(\alpha)|k|^\alpha) & \text{if } k \neq 0 \\ 0 & \text{if } k = 0 \end{cases}$$

where $\zeta(\alpha) := \sum_{k=1}^{\infty} k^{-\alpha}$. Let $\varphi(\theta) = \varphi_\alpha(\theta)$ be the characteristic function of the $\zeta(\alpha)$ -distribution.

(i) For which values of $\alpha > 1$ does the distribution have finite first moment? (ii) Show that if $1 < \alpha \leq 2$ then the behavior of the characteristic function near $\theta = 0$ is as follows:

$$(1) \quad 1 - \varphi_\alpha(\theta) \sim C|\theta|^{\alpha-1}.$$

Notice that this implies that φ_α is not differentiable at $\theta = 0$.

HINT: For $\theta > 0$ write the characteristic function as

$$\varphi_\alpha(\theta) = \frac{1}{\zeta(\alpha)} \sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k^\alpha} = 1 - \theta^{\alpha-1} \frac{\theta}{\zeta(\alpha)} \sum_{k=1}^{\infty} \frac{(1-d)\cos(k\theta)}{(k\theta)^\alpha}.$$

The last sum should look like a Riemann sum for some integral.

(B) One of the many things that Fourier series are good for is evaluating infinite series. In particular, it is possible to evaluate the normalizing constants $\zeta(\alpha)$ for certain integer values of α . Here is how it works for $\alpha = 2$: (i) Find the Fourier coefficients of the function $A(\theta) = \theta$ for $\theta \in [-\pi, \pi]$. (This function has a discontinuity at π (and $-\pi$), so its Fourier coefficients are not absolutely summable; however, they are square-summable.) (ii) Now use the Plancherel formula to evaluate $\zeta(2)$.

In parts (C), (D), and (E) assume that ξ_1, ξ_2, \dots are independent, identically distributed, integer-valued random variables whose distribution is nonlattice and *symmetric*. Let $\varphi(\theta)$ be the common characteristic function, and set $S_n = \sum_{j=1}^n \xi_j$.

(C) Use a modification of Laplace's method to show that if the characteristic function φ satisfies the relation

$$(2) \quad 1 - \varphi(\theta) \sim C|\theta|^\beta$$

for some positive constant C then for some (possibly different) positive constant C' , as $n \rightarrow \infty$,

$$P\{S_n = 0\} \sim \frac{C'}{n^{1/\beta}}.$$

HINTS: First show that the symmetry of the distribution implies that the characteristic function is real-valued and even. Use this together with the Fourier Inversion Theorem to write

$$P\{S_n = 0\} = \frac{1}{\pi} \int_0^\pi \varphi(\theta)^n d\theta.$$

(D) For which values of $\beta \in (0, 1]$ is the random walk S_n recurrent?

(E) Show that the characteristic function of $S_n/n^{1/\beta}$ has a limit as $n \rightarrow \infty$. You might recognize that the limiting characteristic function when $\beta = 1$ is the characteristic function of the *Cauchy* distribution. (You should recall from Stat 304 that convergence of characteristic functions implies that the random variables converge in distribution.)

2. Cauchy distribution and simple random walk on \mathbb{Z}^2 : The *simple random walk* on the integer lattice \mathbb{Z}^2 in two dimensions is the Markov chain whose transition probabilities are such that the next state is always chosen at random from the four nearest neighbors of the current state. Denote the state at time n by $S_n = (S_n^X, S_n^Y)$. For any integer $m \geq 1$ define

$$\tau_m = \min\{n \geq 1 : S_n^Y = m\}$$

to be the time of the first visit to the line $y = m$. (This is finite because the simple random walk in 2D is recurrent.) Define $W_0 = 0$ and

$$W_m = S_{\tau_m}^X$$

to be the value of the X -coordinate at the time when the random walk first visits the line $y = m$.

(A) Show that the increments of the sequence $\{W_m\}_{m \geq 0}$ are independent and identically distributed.

(B) Find the characteristic function $\psi(\theta)$ of W_1 . HINT: Condition on the first step of the random walk. If the first step is to $(0, -1)$ then the random walk must cross *two* horizontal levels to reach the line $y = 1$.

(C) Show that for some constant $\gamma > 0$ the sequence of random variables W_m/m^γ converges in distribution as $m \rightarrow \infty$. What is γ , and what is the limit distribution? HINT: Characteristic functions are your friends.

(D) Show that the random walk $\{W_m\}_{m \geq 0}$ is recurrent. Is it positive recurrent or null recurrent?

3. Fourier analysis on finite abelian groups. One of the big theorems of abstract algebra has it that every finite abelian group is isomorphic to $\mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_k}$, where \mathbb{Z}_m is the integers mod m . (That is, every finite abelian group is a finite k -dimensional *torus*.) There is a simple version of Fourier analysis for any such group. Here's how it goes for \mathbb{Z}_m (the *discrete circle* with m points):

(A) For each $k \in \mathbb{Z}_m$ (that is, for each integer $k = 0, 1, 2, \dots, m - 1$), define $\delta_k : \mathbb{Z}_m \rightarrow \mathbb{C}$ by

$$\begin{aligned} \delta_k(j) &= 1 & \text{if } k = j; \\ &= 0 & \text{if } k \neq j. \end{aligned}$$

Check that the functions δ_k are orthonormal (relative to the counting measure on \mathbb{Z}_m). NOTE: This should be an entirely trivial calculation. The functions δ_k are the *standard orthonormal basis* for $L^2(\mathbb{Z}_m)$.

(B) For each integer $k = 0, 1, 2, \dots, m - 1$ define a function $e_k : \mathbb{Z}_m \rightarrow \mathbb{C}$ by

$$e_k(j) = e^{2\pi i j k} \quad \text{for } j \in \mathbb{Z}_m.$$

Check that

$$\begin{aligned} \sum_{j \in \mathbb{Z}_m} e_k(j) \bar{e}_l(j) &= 0 \quad \text{if } l \neq k \\ &= m \quad \text{if } l = k. \end{aligned}$$

This implies that the functions e_k/\sqrt{m} are an orthonormal basis of $L^2(\mathbb{Z}_m)$. This is the *Fourier basis*.

(C) For any function $f : \mathbb{Z}_m \rightarrow \mathbb{C}$ (i.e., any $f \in L^2(\mathbb{Z}_m)$) define its *Fourier transform* to be the function $\hat{f} : \mathbb{Z}_m \rightarrow \mathbb{C}$ given by

$$\hat{f}(k) := \sum_{j \in \mathbb{Z}_m} f(j) \bar{e}_k(j).$$

Formulate and prove versions of the *Plancherel theorem* and the *Fourier inversion formula* for this transform. The first should express the fact that the Fourier transform is an L^2 -linear isometry, and the second that the function f can be recovered from its Fourier transform by inverse Fourier transform. Note: There has to be a normalizing factor of $1/m$ or $1/\sqrt{m}$ (you figure out what it should be!) to account for the fact that counting measure on \mathbb{Z}_m isn't a probability measure.

(D) Define a suitable notion of *convolution* of two functions on \mathbb{Z}_m , and prove that Fourier transform takes convolutions to products.

(E) Now consider *simple random walk* S_n on the group \mathbb{Z}_m : this is the Markov chain that evolves by moving one step either clockwise or counter-clockwise, with probability $1/2$ each, on \mathbb{Z}_m (viewed as a discrete circle). Assume that $S_0 = 0$. Show that

$$P\{S_n = j\} = \frac{1}{m} \sum_{k=0}^m \varphi^n(k) \bar{e}_k(j)$$

where $\varphi(k)$ is (you figure it out). Use this to show that the distribution of S_n approaches the uniform distribution on \mathbb{Z}_m as $n \rightarrow \infty$.

(F)* Assume that m is large. How large must n be, relative to m , in order that the total variation distance between the distribution of S_n and the uniform distribution is $< 1/4$? NOTE: This is the simplest instance of how one uses *spectral analysis* to determine the mixing rate for a finite state Markov chain. A good deal of modern research has taken place on how to use *group representation theory* (the nonabelian version of Fourier analysis) to determine the mixing rates of random walks on various finite nonabelian groups. See P. DIACONIS, *Group representations in probability and statistics* for an introduction to this subject.

Haven't had enough yet?

4. Hitting probabilities in $d \geq 3$ dimensions. Let S_n be the position at time n of a simple random walk on the d -dimensional integer lattice \mathbb{Z}^d . In $d \geq 3$ the random walk is transient, and so if the starting point of the random walk is $x \neq 0$ (under P^x) then there is positive probability that it will never visit the origin. The aim of this problem is to show how to estimate this probability for starting positions x that are far away from the origin 0.

(A) Let $\varphi(\theta)$ be the characteristic function of the random walk, that is,

$$\varphi(\theta) = d^{-1} \sum_{j=1}^d \cos \theta_j \quad \text{where } \theta = (\theta_1, \theta_2, \dots, \theta_d).$$

Prove that under P^x (i.e., when the starting point of the random walk is x) the expected number of visits $G(x, 0)$ to the origin is

$$G(x, 0) := E^x \sum_{n=0}^{\infty} \mathbf{1}\{S_n = 0\} = (2\pi)^{-d} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{e^{i\langle x, \theta \rangle}}{1 - \varphi(\theta)} d\theta.$$

(Here $\langle x, \theta \rangle$ denotes the inner product of the vectors x, θ .) Note: The function $G(x, 0)$ is called the *Green's function* of the random walk. Hint: First show that for every $0 < t < 1$,

$$\sum_{n=0}^{\infty} t^n P^x \{S_n = 0\} = (2\pi)^{-d} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{e^{i\langle x, \theta \rangle}}{1 - t\varphi(\theta)} d\theta.$$

Then let $t \uparrow 1$ and use the dominated convergence theorem.

(B) Use the strong Markov property to show that

$$G(x, 0) = P^x \{S_n = 0 \text{ for some } n \geq 0\} G(0, 0).$$

(C)** Use the result of part (A) to show that there is a constant $C > 0$ such that as $|x| \rightarrow \infty$,

$$G(x, 0) \sim C|x|^{-d+2}$$

HINT: First do the substitution $\alpha = |x|\theta$ in the Fourier integral. Then partition the resulting integral into two parts: first, where $|\alpha| > A$ and second, where $|\alpha| \leq A$. Show that (i) when A is large the contribution of the integral over the region $|\alpha| > A$ is small; and (ii) that for each A the integral over $\alpha \leq A$ converges as $|X| \rightarrow \infty$, and that as $A \rightarrow \infty$ the limits converge.

Note: The traditional way of approaching (C) is to use a strong form of the *local central limit theorem*. See Spitzer's book *Principles of Random Walk* to see this done in detail.