THE RENEWAL THEOREM

1. Renewal Processes

1.1. **Example: A Dice-Rolling Problem.** Before the days of video games, kids used to pass their time playing (among other things) board games like Monopoly, Clue, Parcheesi, and so on. In all of these games, players take turns rolling a pair of dice and then moving tokens on a board according to rules that depend on the dice rolls. The rules of these games are all somewhat complicated, but in most of the games, in most circumstances, the number of spaces a player moves his/her token is the sum of the numbers showing on the dice.

Consider a simplified board game in which, on each turn, a player rolls a single fair die and then moves his/her token ahead X spaces, where X is the number showing on the die. Assume that the board has K spaces arranged in a cycle, so that after a token is moved ahead a total of K spaces it returns to its original position. Assign the spaces labels $0,1,2,\dots,K-1$ in the order they are arranged around the cycle, with label 0 assigned to the space where the tokens are placed at the beginning of the game (in Monopoly, this would be the space marked "Go"). Then the movement of a player's token may be described as follows: after n turns the token will be on the space labelled $S_n \mod K$, where

$$S_n = \sum_{i=1}^{n} X_i \tag{1}$$

and $X_1, X_2, \dots$ are the dice rolls obtained on successive turns.

Problem 0: What is the probability that on the tenth time around the board a player's token lands on the space labelled $K-1$?

Anyone who has played Monopoly will know that this particular space is "Boardwalk", and that if another player owns it then it is a bad thing to land on it, especially late in the game. Of course, the answer to Problem 0 isn't especially relevant to Monopoly, because the rules of movement aren't the simple rules we have imposed above; in fact, the problem is really more interesting in Monopoly, because there it turns out that the probability of landing on Boardwalk isn't quite the same as that of landing on (say) Park Place (space $K-3$). But we have to start somewhere, so Problem 0 is what we'll consider. Here is a more general problem, formulated in a more convenient way:

Problem 1: Fix an integer $x \geq 0$. What is the probability u_x that $S_n = x$ for some n?

Observe that when $x = 10K-1$, this reduces to Problem 0.

There is a related issue in Monopoly that is of some interest: when you pass Go, there is a block of spaces (Baltic and Mediterranean Avenues) that you might want to avoid if you don't own them. So what is the probability that on the tenth trip around the board you manage to jump over them altogether (that is, the chance that you don't land on any of the spaces labelled 1,2,3)? Here is a more general formulation:

Problem 2: Fix $t \geq 0$, and define the *first-passage* and *overshoot* random variables by

$$\tau(t) = \min\{n \geq 0 : S_n > t\} \quad \text{and} \tag{2}$$

$$R(t) = S_{\tau(t)} - t \tag{3}$$

What is the probability distribution of $R(t)$?

1.2. **Renewal Processes.** Problems 1 and 2 above are prototypical problems in *renewal theory*. In general, a *renewal process* is the increasing sequence of random nonnegative numbers $0, S_1, S_2, \dots$ visited by a random walk $S_n = \sum_{i=1}^n X_i$ that only makes *positive* steps. The individual terms S_n of this sequence are called *renewals*, or sometimes *occurrences*. With each renewal process is associated a *renewal counting process* $N(t)$ that tracks the total number of renewals (not including the initial occurrence) to date: the random variable $N(t)$ is defined by

$$N(t) = \max\{n : S_n \leq t\}. \tag{4}$$

In many instances (but not the dice-rolling example) the renewals have a natural interpretation as the *times* at which some irregularly occurring event take place. For example, imagine that $X_1, X_2, \dots$ are the lifetimes of replaceable components in a system, such as light bulbs in a light socket. If the first component is inserted at time 0, then the renewals S_n are the times at which components must be replaced. It is traditional in the subject to refer to the random variables X_i as *interoccurrence times*, and their common distribution as the *interoccurrence time distribution*. The random variables

$$A(t) = t - S_{N(t)}, \tag{5}$$

$$R(t) = S_{\tau(t)} - t, \quad \text{and} \tag{6}$$

$$L(t) = S_{\tau(t)} - S_{N(t)} \tag{7}$$

are usually called the *age*, *residual lifetime*, and *total lifetime* random variables (although in certain applications it is also common to refer to $R(t)$ as the *overshoot*). Note that $L(t) = A_t + R(t)$.

1.3. **Poisson and Bernoulli Processes.** Two special cases are especially important. When the interoccurrence time distribution is the exponential distribution with mean $1/\lambda$, then the corresponding renewal process is the *Poisson process* with intensity λ. When the interoccurrence time distribution is the geometric distribution with parameter p, that is,

$$P\{X_i = k\} = p(1-p)^{k-1} \quad \text{for } k = 1, 2, \dots,$$

then the corresponding renewal process is the *Bernoulli process* with success parameter p. Many interesting problems that are quite difficult for renewal processes in general have simple solutions for Poisson and Bernoulli processes. For instance, in general the distribution of the count random variable $N(t)$ does not have a simple closed form, but for a Poisson process it has the Poisson distribution with mean λt. Similarly, the age and residual lifetime random variables $A(t)$ and $R(t)$ generally have distributions that do not depend on the interoccurrence time distribution in a simple way, but for the Poisson process they are, for any fixed t, just independent exponential random variables with mean $1/\lambda$. These properties follow easily from the memoryless property of the exponential distribution. Renewal processes are important because in many systems the times between successive renewals do not have memoryless distributions.

Exercise: What are the distributions of A_t, R_t, and L_t for a Bernoulli process?

2. RENEWAL THEORY: THE LATTICE CASE

Renewal theory breaks into two parallel streams, one for discrete time, the other for continuous time. These are called the *lattice* and *nonlattice* cases. A probability distribution is said to be *nonlattice* if it does not attach all of its mass to an arithmetic progression $h\mathbb{Z}$ consisting of all integer multiples of a constant $h > 0$ constant called the *span* of the progression. Conversely, a probability distribution is said to be *lattice* if it is entirely supported by an arithmetic progression $h\mathbb{Z}$. The lattice case is mildly complicated by the fact that a probability distribution that is supported by an arithmetic progression $h\mathbb{Z}$ might actually be entirely supported by a smaller arithmetic progression, for instance $2h\mathbb{Z}$. It is an easy exercise in elementary number theory to show that, with the exception of the probability distribution that puts mass 1 at the single point 0, any lattice distribution is supported by a unique minimal arithmetic progression $h\mathbb{Z}$. The value of h is called the *period* of the distribution. There is no real loss of generality in restricting attention to interoccurrence time distributions with period $h = 1$, and so henceforth we shall assume

$$p_x = P\{X_i = x\} \quad \text{for} \quad x = 1, 2, 3, \dots \tag{8}$$

that are supported by the positive integers, but not by an arithmetic progression $k\mathbb{Z}$ with $k \geq 2$. Throughout this section, therefore, we make the following assumptions:

Assumption 1. The interoccurrence time distribution $\{p_x\}_{x \in \mathbb{N}}$ is supported by the positive integers, and for every integer $k \geq 2$,

$$\sum_{n=1}^{\infty} p_{nk} < 1. \tag{9}$$

Assumption 2. The interoccurrence time distribution has finite mean

$$\mu = \sum_{x=1}^{\infty} x p_x < \infty. \tag{10}$$

2.1. **Erdös-Feller-Pollard Theorem.** The cornerstone of renewal theory in the lattice case is the *renewal theorem* of Erdös, Feller, and Pollard. Let $0 = S_0, S_1, S_2, \dots$ be a renewal process whose interoccurrence time distribution $\{p_x\}$ satisfies Assumptions 1–2. Define the *renewal measure* to be the sequence

$$\begin{aligned} u_x &= \sum_{n=0}^{\infty} P\{S_n = x\} \\ &= P\{\text{there is a renewal at time } x\} \quad \text{for } x = 0, 1, 2, \dots \end{aligned} \tag{11}$$

Erdös- Feller- Pollard Renewal Theorem: $\lim_{x \to \infty} u_x = 1/\mu$.

Feller, Erdös, and Pollard gave several different proofs of their theorem, and several more have since been found. One of these (discovered by one of my favorite probabilists in 1988) is outlined below. First, though, we shall look at an interesting consequence of the theorem.

2.2. **Limiting Age Distribution.** Recall that A_t is the age of the component in use at time t. In order that $A(t)=k$ for nonnegative integers t,k, it must be the case that the last renewal before time t took place at time $t-k$, equivalently, that there was a renewal at time $t-k$ *and* that the lifetime of the component installed at that time exceeds k. These two events are independent (Exercise: Why?), so

$$P\{A(t)=k\}=u_{t-k}P\{X_1>k\}=u_{t-k}\sum_{x=k+1}^{\infty}p_x.$$

But the Feller-Erdös-Pollard theorem implies that $u_{t-k}\to 1/\mu$ as $t\to\infty$. This proves

Corollary 1. *For each* $k=0,1,2,\dots,$

$$\lim_{t\to\infty}P\{A(t)=k\}=\mu^{-1}\sum_{x=k+1}^{\infty}p_x. \tag{12}$$

Exercise: Check that the right side defines a probability distribution.

3. Proof of the Feller-Erdös-Pollard Theorem

The Feller-Erdös-Pollard theorem asserts that under Assumptions 1-2 of section 2 the renewal measure u_m converges to $1/\mu$ as $m\to\infty$. In this section I will give a proof of this assertion under the following more restrictive hypotheses on the interoccurrence time distribution p_k:

Assumption 3. There is a finite integer K such that $\sum_{k=1}^{K}p_k=1$.

Assumption 4. $p_1>0$.

3.1. **The Elementary Renewal Theorem.** There are two issues in the proof: first, we must show that the limit $\lim_{x\to\infty}u_x$ exists, and second, that the limit is $1/\mu$. The second point – the identification of the limit – will turn on another important result in renewal theory, sometimes called (especially in the older literature) the *Elementary Renewal Theorem.* Recall that $N(k)$ denotes the number of renewals by time k.

Proposition 2. *(Elementary Renewal Theorem)*

$$\lim_{k\to\infty}\frac{EN(k)}{k}=\frac{1}{\mu}. \tag{13}$$

Proof. Let $\tau(k)=\min\{n\ge 0: S_n\ge k\}$. Since the interoccurrence times are never less than one, there can be at most one renewal at any given time, and so the number of renewals $N(k)$ by time k is the same as the number of steps $\tau(k)$ needed by the random walk to reach the interval $[k,\infty)$; thus, $\tau(k)=N(k)$. Clearly, $\tau(k)$ is a *stopping time* for the random walk S_n, and so Wald's First Identity will apply, provided we can show that $E\tau(k)<\infty$. But each step X_i is of size at least one, so the number of steps needed to reach the interval $[k,\infty)$ cannot be larger than k; hence, $\tau(k)\le k$ and so $E\tau(k)\le k$. Wald's identity implies that

$$ES_{\tau(k)}=\mu E\tau(k)=\mu EN_k.$$

Now consider the random variable $S_{\tau(k)}$: this is where the random walk lands when it first enters the interval $[k,\infty)$. By Assumption 3, the random walk never makes a step of size more than K, so we must have $k\le S_{\tau(k)}\le k+K$. Therefore, for any $k=1,2,\dots,$

$$k\le \mu EN(k)\le k+K.$$

Dividing by k and taking limits as $k \to \infty$, we obtain the desired result. $\square$

3.2. **Identification of the limit.** Let's begin the proof of the Erdös-Feller-Pollard theorem by showing that if the limit $\alpha = \lim_{n\to\infty} u_n$ exists, then it must equal $1/\mu$. If limit $\alpha = \lim_{n\to\infty} u_n$ exists, then the rolling averages of the numbers $u_1, u_2, \dots$ must also converge to α, that is,

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m} u_m = \alpha.$$

Now recall that u_m is the probability that the random walk $(S_{n\,n\geq 0})$ ever visits the point m; consequently,

$$\begin{aligned}\sum_{i=0}^{m} u_m &= \sum_{i=0}^{m} P\{\text{renewal at } m\} \\ &= E\sum_{i=0}^{m} \mathbf{1}\{\text{renewal at } m\} \\ &= EN(m)\end{aligned}$$

But by the elementary renewal theorem, $EN(m)/m$ converges to $1/\mu$ as $m \to \infty$. It follows that $\alpha = 1/\mu$.

3.3. **Existence of the limit.** Fix an interoccurrence time distribution $F = \{p_k\}_{k\in\mathbb{N}}$, and let $X_1, X_2, \dots$ be independent, identically distributed random variables with distribution F. For any integer x and $n = 0, 1, 2, \dots$, define

$$S_n^x = x + S_n \quad \text{where} \quad S_n = X_1 + X_2 + \cdots + X_n. \tag{14}$$

Lemma 3. *For any integer $x \geq 1$,*

$$u_x = \sum_{n=0}^{\infty} P\{S_n^{-x} = 0\} \tag{15}$$

Proof. The event $\{S_n^{-x} = 0\}$ coincides with the event $\{S_n = x\}$, and so the two events have the same probability. Consequently, (15) follows from the definition (**??**). $\square$

The proof that the limit $\lim_{m\to\infty} u_m$ exists will rest on a simultaneous construction of different versions of the processes S_n^{-x}. This construction uses a system of random "arrows" attached to the integers $x \leq -1$. For this purpose, let $\cdots Y_1, Y_0, Y_{-1}, Y_{-2}, \dots$ be independent, identically distributed random variables with common distribution F. Imagine that for each integer x there is an arrow that points from x to $x + Y_x$. Now for each starting point x, build a path γ_n^x by starting at $x = \gamma_0^x$ and following the arrows:

$$\gamma_{n+1}^x = \gamma_n^x + Y_{\gamma_n^x}. \tag{16}$$

Since each new arrow encountered along the way is chosen independently of all the previous arrows, the random path $\{\gamma_n^x\}_{n\geq 0}$ has the same joint distribution as $\{S_n^x\}_{n\geq 0}$. Therefore, this path has the same chance of passing through the point 0, which by Lemma 3 is u_{-x}. This proves

Corollary 4. *For each integer $x \leq -1$,*

$$u_{-x} = P\{\gamma_n^x = 0 \textit{ for some } n \geq 0\}. \tag{17}$$

Next, we consider two paths γ^x, γ^y with different starting points $x, y \le -1$. It is possible that these paths might visit the same point z: for instance, if $x = y-1$ and $Y_x = Y_y + 1$ then this will happen after the very first step. But note: If the paths γ^x and γ^y ever do visit the same point z, then *they must follow the same arrows subsequently.* Hence, if they visit a common point $z \le -1$ then either *both* must eventually pass through the point 0 or *neither* will.

Corollary 5. *For any two integers $x, y \le -1$, define $G_{x,y}$ to be the event that the random paths γ^x and γ^y meet (and therefore coalesce) at some point $z \le -1$. Then*

$$|u_{-x} - u_{-y}| \le 1 - P(G_{x,y}) \tag{18}$$

Proof. By Corollary 4,

$$|u_{-x} - u_{-y}| = |P\{\gamma^x \text{ visits } 0\} - P\{\gamma^y \text{ visits } 0\}| \le 1 - P(G_{x,y}).$$

□

Now we will show that if $-x, -y$ are both large, then $P(G_{x,y})$ is nearly 1. This is where Assumptions 3-4 come in. Suppose that somewhere to the left of the point -1 there is a block $\mathscr{B}$ of K consecutive points $z-K, z-K+1, \dots z-1$ where all of the arrows are of size $+1$. Then all paths γ^x that start at points $x < z-K$ must pass through the point z! This is because by Assumption 3 no arrows are longer than K, and so every path γ^x such that $x < z-K$ must hit some point in the block $\mathscr{B}$; but every such path must then proceed in jumps of size 1 until reaching z.

Define a *bottleneck* to be block $\mathscr{B}$ of K consecutive points $z-K, z-K+1, \dots z-1$ where all of the arrows are of size $+1$.

Corollary 6. *For any two integers $x < y \le -1$, the probability $P(G_{x,y})$ that the paths γ^x, γ^y meet at some $z \le -1$ is at least as large as the probability that there is a bottleneck between y and -1.*

In order that the points $z-K, z-K+1, \dots z-1$ form a bottleneck, the arrows attached to these points must all be of size 1. By Assumption 4, the probability of this is $p_1^K = \beta > 0$. Now between the points y and -1 there are $-y/K - 1$ nonoverlapping consecutive blocks of K successive integers. Since the arrows attached to the points in these blocks are independent, it follows that the probability that there is no bottleneck between y and -1 is no larger than $(-1\beta)^{-y/K-1}$. This obviously converges to 0 as $y \to -\infty$. Thus, by Corollaries 6 and 5, we have proved

Corollary 7. *The sequence u_x converges to a limit as $x \to \infty$.*

This completes the proof of the Erdös-Feller-Pollard theorem under the Assumptions 3 and 4. Are these assumptions *necessary?* No!

Problem 1. By modifying the argument above, show that Assumption 4 is unnecessary. HINT: You will need to show that there is a local configuration of arrows that forms a "bottleneck". This will be a bit more complicated than the simple block of K consecutive arrows of length 1.

Problem 2. (Much harder) Now show that Assumption 3 is unnecessary. HINT: The key is the hypothesis that the step distribution must have finite mean μ. This guarantees that

$$\sum_{k=1}^{\infty} P\{X_1 \ge k\} < \infty.$$

(Why?) It then follows by the *Borel-Cantelli Lemma* that if $X_1, X_2, \dots$ all have the same distribution, then with probability one only finitely many of the events $\{X_k \ge k\}$ can occur.

4. Renewal Equation and Generating Functions

4.1. **Renewal equation for the renewal measure.** Recall that the *renewal measure* is the sequence u_m = probability that there is a renewal at time m, or, equivalently, u_m =probability that the random walk $(S_n)_{n\geq 0}$ ever visits m. For any integer $m \geq 0$, there are two possibilities: either (i) $m = 0$, in which case $u_0 = 1$, because the random walk $(S_{n\,n\geq 0})$ starts at $S_0 = 0$; or (ii) $m \geq 1$, in which case the random walk $(S_{n\,n\geq 0})$ must make at least one step before visiting m. In order that a visit to $m \geq 1$ can occur, the first step $X_1 = S_1$ of the random walk must be to one of the sites $k = 1, 2, 3, \dots, m$, because if $S_1 > m$ then all future locations S_n will be to the right of m. Therefore,

$$u_m = \delta_{0,m} + \sum_{k=1}^{m} p_k u_{m-k} \tag{19}$$

where

$$p_k = P\{X_1 = k\}.$$

Here $\delta_{0,m}$ is the *Kronecker delta* sequence, that is, $\delta_{0,0} = 1$ and $\delta_{0,m} = 0$ for all $m \neq 0$. Equation (19) is known as the *renewal equation* for the sequence u_m.

4.2. **Generating functions.** In certain cases, *exact* solutions of the renewal equation can be obtained. These are found using *generating functions*.

Definition 1. The *generating function* of a sequence $\{a_n\}_{n\geq 0}$ of real (or complex) numbers is the function $A(z)$ defined by the power series

$$A(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{20}$$

Observe that for an arbitrary sequence $\{a_n\}$ the series (20) need not converge for all complex values of the argumetn z. In fact, for some sequences the series (20) diverges for *every* z except $z = 0$: this is the case, for instance, if $a_n = n^n$. But for many sequences of interest, there will exist a positive number R such that, for all complex numbers z such that $|z| < R$, the series (20) converges absolutely. In such cases, the generating function $A(z)$ is said to have positive *radius of convergence*. The generating functions in all of the problems considered in these notes will have positive radius of convergence. Notice that if the entries of the sequence a_n are probabilities, that is, if $0 \leq a_n \leq 1$ for all n, then the series (20) converges absolutely for all z such that $|z| < 1$.

If the generating function $A(z)$ has positive radius of convergence then, at least in principal, all information about the sequence $\{a_n\}$ is encapsulated in the generating function $A(z)$. In particular, each coefficient a_n can be recovered from the function $A(z)$, since $n!a_n$ is the nth derivative of $A(z)$ at $z = 0$. Other information may also be recovered from the generating function: for example, if the sequence $\{a_n\}$ is a discrete probability density, then its mean may be obtained by evaluating $A'(z)$ at $z = 1$, and all of the higher moments may be recovered from the higher derivatives of $A(z)$ at $z = 1$.

A crucially important property of generating functions is the *multiplication law*: The generating function of the convolution of two sequences is the product of their generating functions.

Exercise 1. Prove this.

The multiplication law is the basis of most uses of generating functions in random walk theory, and all of the examples considered below. Note that for *probability* generating functions, this fact is a consequence of the multiplication law for expectations of independent random variables: If X and Y are independent, nonnegative-integer valued random variables, then

$$Ez^{X+Y} = Ez^X Ez^Y. \tag{21}$$

The renewal equation (19) translates into a simple algebraic equation relating the generating functions of the sequences u_m and p_k. Define

$$U(z) = \sum_{m=0}^{\infty} u_m z^m \quad \text{and}$$
$$F(z) = \sum_{k=1}^{\infty} p_k z^k = Ez^{X_1}.$$

Multiplying both sides of the renewal equation by z^m, then summing over m and using the multiplication law for convolution yields

$$U(z) = 1 + F(z)U(z) \quad \Longrightarrow \tag{22}$$

$$\boxed{U(z) = \frac{1}{1 - F(z)}} \tag{23}$$

4.3. **Partial Fraction Decompositions.** Formula (23) tells us how the generating function of the renewal sequence is related to the probability generating function of the steps X_j. Extracting useful information from this relation is, in general, a difficult analytical task. However, in the special case where the probability distribution $\{f_k\}$ has finite support, the method of *partial fraction decomposition* provides an effective method for recovering the terms u_m of the renewal sequence. Observe that when the probability distribution $\{f_k\}$ has finite support, its generating function $F(z)$ is a *polynomial*, and so in this case the generating function $U(z)$ is a *rational function*.[1]

The strategy behind the method of partial fraction decomposition rests on the fact that a *simple pole* may be expanded as a geometric series: in particular, for $|z| < 1$,

$$(1 - z)^{-1} = \sum_{n=0}^{\infty} z^n. \tag{24}$$

Differentiating with respect to z repeatedly gives a formula for a pole of order $k + 1$:

$$(1 - z)^{-k-1} = \sum_{n=k}^{\infty} \binom{n}{k} z^{n-k}. \tag{25}$$

Suppose now that we could write the generating function $U(z)$ as a sum of poles $C/(1-(z/\zeta))^{k+1}$ (such a sum is called a *partial fraction decomposition*). Then each of the poles could be expanded in a series of type (24) or (25), and so the coefficients of $U(z)$ could be obtained by adding the corresponding coefficients in the series expansions for the poles.

Example: Consider the probability distribution $f_1 = f_2 = 1/2$. The generating function F is given by $F(z) = (z + z^2)/2$. The problem is to obtain a partial fraction decomposition for $(1-$

[1] A *rational function* is the ratio of two polynomials.

$F(z))^{-1}$. To do this, observe that at every pole $z = \zeta$ the function $1 - F(z)$ must take the value 0. Thus, we look for potential poles at the zeros of the polynomial $1 - F(z)$. In the case under consideration, the polynomial is quadratic, with roots $\zeta_1 = 1$ and $\zeta_2 = -2$. Since each of these is a simple root both poles should be simple; thus, we should try

$$\frac{1}{1-(z+z^2)/2} = \frac{C_1}{1-z} + \frac{C_2}{1+(z/2)}.$$

The values of C_1 and C_2 can be gotten either by adding the fractions and seeing what works or by differentiating both sides and seeing what happens at each of the two poles. The upshot is that $C_1 = 2/3$ and $C_2 = 1/3$. Thus,

$$U(z) = \frac{1}{1-F(z)} = \frac{2/3}{1-z} + \frac{1/3}{1+(z/2)}. \tag{26}$$

We can now read off the renewal sequence u_m by expanding the two poles in geometric series:

$$u_m = \frac{2}{3} + \frac{1}{3}(-2)^m. \tag{27}$$

There are several things worth noting. First, the renewal sequence u_m has limit 2/3. This equals $1/\mu$, where $\mu = 3/2$ is the mean of the distribution $\{f_k\}$. We should be reassured by this, because it is what the Feller-Erdös-Pollard Renewal Theorem predicts the limit should be. Second, the remainder term $(1/3)(-2)^{-m}$ decays exponentially in m. As we shall see, this is always the case for distributions $\{f_k\}$ with finite support. It is not always the case for arbitrary distributions $\{f_k\}$, however.

Problem 3. The *Fibonacci sequence* $1, 1, 2, 3, 5, 8, \dots$ is the sequence a_n such that $a_1 = a_2 = 1$ and such that

$$a_{m+2} = a_m + a_{m+1}.$$

(A) Find a functional equation for the generating function of the Fibonacci sequence. (B) Use the method of partial fractions to deduce a formula for the terms of the Fibonacci sequence. NOTE: Your answer should involve the so-called *golden ratio* (the larger root of the equation $x^2 - x - 1 = 0$.)

4.4. **Step Distributions with Finite Support.** Assume now that the step distribution $\{f_k\}$ has finite support, is nontrivial (that is, does not assign probability 1 to a single point) and is nonlattice (that is, it does not give probability 1 to a proper arithmetic progression). Then the generating function $F(z) = \sum f_k z^k$ is a polynomial of degree at least two. By the Fundamental Theorem of Algebra, $1 - F(z)$ may be written as a product of linear factors:

$$1 - F(z) = C \prod_{j=1}^{K} (1 - z/\zeta_j) \tag{28}$$

The coefficients ζ_j in this expansion are the (possibly complex) roots of the polynomial equation $F(z) = 1$. Since the coefficients f_k of $F(z)$ are real, the roots of $F(z) = 1$ come in conjugate pairs; thus, it is only necessary to find the roots in the upper half plane (that is, those with nonnegative imaginary part). In practice, it is usually necessary to solve for these roots numerically. The following proposition states that none of the roots is *inside* the unit circle in the complex plane.

Lemma 8. *If the step distribution $\{f_k\}$ is nontrivial, nonlattice, and has finite support, then the polynomial $1 - F(z)$ has a simple root at $\zeta_1 = 1$, and all other roots ζ_j satisfy the inequality $|\zeta_j| > 1$.*

Proof. It is clear that $\zeta_1 = 1$ is a root, since $F(z)$ is a probability generating function. To see that $\zeta_1 = 1$ is a *simple* root (that is, occurs only once in the product (28)), note that if it were a multiple root then it would have to be a root of the derivative $F'(z)$ (since the factor $(1-z)$ would occur at least twice in the product (28)). If this were the case, then $F'(1) = 0$ would be the mean of the probability distribution $\{f_k\}$. But since this distribution has support $\{1,2,3\cdots\}$, its mean is at least 1.

In order that ζ be a root of $1-F(z)$ it must be the case that $F(\zeta) = 1$. Since $F(z)$ is a probability generating function, this can only happen if $|\zeta| \geq 1$. Thus, to complete the proof we must show that there are no roots of modulus one other than $\zeta = 1$. Suppose, then, that $\zeta = e^{i\theta}$ is such that $F(\zeta) = 1$, equivalently,

$$\sum f_k e^{i\theta k} = 1.$$

Then for every k such that $f_k > 0$ it must be that $e^{i\theta k} = 1$. This implies that θ is an integer multiple of $2\pi/k$, and that this is true for every k such that $f_k > 0$. Since the distribution $\{f_k\}$ is nonlattice, the greatest common divisor of the integers k such that $f_k > 0$ is 1. Hence, θ is an integer multiple of 2π, and so $\zeta = 1$. □

Corollary 9. *If the step distribution $\{f_k\}$ is nontrivial, nonlattice, and has finite support, then*

$$\frac{1}{1-F(z)} = \frac{1}{\mu(1-z)} + \sum_{r=1}^{R} \frac{C_r}{(1-z/\zeta_r)^{k_r}} \tag{29}$$

where μ is the mean of the distribution $\{f_k\}$ amd the poles ζ_r are all of modulus strictly greater than 1.

Proof. The only thing that remains to be proved is that the simple pole at 1 has residue $1/\mu$. To see this, multiply both sides of equation (29) by $1-z$:

$$\frac{1-z}{1-F(z)} = C + (1-z)\sum_r \frac{C_r}{(1-z/\zeta_r)^{k_r}}.$$

Now take the limit of both sides as $z \to 1-$: the limit of the right side is clearly C, and the limit of the left side is $1/\mu$, because μ is the derivative of $F(z)$ at $z = 1$. Hence, $C = 1$. □

Corollary 10. *If the step distribution $\{f_k\}$ is nontrivial, nonlattice, and has finite support, then*

$$\lim_{m\to\infty} u_m = 1/\mu, \tag{30}$$

and the remainder decays exponentially as $m \to \infty$.

Remark. The last corollary is, of course, a special case of the Feller-Erdös-Pollard Renewal Theorem.