STATISTICS 312: STOCHASTIC PROCESSES
HOMEWORK ASSIGNMENT 6
DUE WEDNESDAY NOVEMBER 16

In the following problems the sequence $(Z_n)_{n\geq 0}$ is assumed to be a Galton-Watson process with $Z_0 = 1$ and offspring distribution

$$P\{Z_1 = k\} = p_k \quad \text{for all } k = 0, 1, 2, \dots .$$

Denote by $\varphi(t) = \sum_{k=0}^{\infty} p_k t^k$ the probability generating function of the offspring distribution.

Problem 1. Total Progeny. Let $S = \sum_{n=1}^{\infty} Z_n$ be the total number of particles ever born in the Galton-Watson process. (Note: this does *not* include the initial particle in generation $n = 0$.) Let $\psi(t) = \sum_{m=0}^{\infty} t^m P\{S = m\}$ be the probability generating function of S. (Note: The sum does *not* include any term for the event $\{S = \infty\}$, even though this event would have positive probability if the Galton-Watson is supercritical.)

(A) Show that

$$\psi(t) = \varphi(t\psi(t)).$$

(B) Consider the special case where the offspring distribution is $p_0 = p_2 = \frac{1}{2}$ and $p_k = 0$ for all $k \neq 0$. Solve the equation in (A) for the generating function $\psi(t)$ and then use Newton's binomial formula to write an explicit formula for

$$P\{S = m\}.$$

Problem 2. Geometric offspring distribution. Suppose that the offspring distribution is *geometric*, that is, for some $0 < \theta < 1$,

$$P\{\xi = k\} = (1 - \theta)\theta^k \quad \text{for all } k = 0, 1, 2, \dots .$$

(A) Verify that the probability generating function $\varphi(z)$ is a *linear fractional transformation.* A linear fractional transformation is a function of the form

$$\varphi(z) = \frac{az + b}{cz + d}.$$

(B) Verify that if

$$\varphi(z) = \frac{az + b}{cz + d} \quad \text{and} \quad \psi(z) = \frac{a'z + b'}{c'z + d'}$$

are two linear fractional transformations then their composition is also a linear fractional transformation; in particular, show that

$$\varphi(\psi(z)) = \frac{a''z + b''}{c''z + d''}$$

where the coefficients a'', b'', c'', d'' are gotten by the matrix multiplication rule

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

NOTE: This can be checked by direct substitution — nothing but some unpleasant algebra is required for the solution. But there is another way to understand it that makes it less mysterious: matrix multiplication maps straight lines in $\mathbb{R}^2$ through the origin to straight lines in $\mathbb{R}^2$ through the origin. Each line (except the horizontal line!) goes through a unique point $(x, 1)$. Identifying the straight line of slope $1/x$ with the point $(x, 1)$ gets you from matrix multiplication to the linear fractional transformation.

(C) Use the result of (B) to identify the distribution of Z_n. HINT: It isn't that hard to raise a 2×2 matrix to the nth power. Diagonalization might help. In cases where there is a double root to the characteristic equation (e.g., $\theta = 1/2$), you could use Jordan canonical form, or just experiment a bit.

Problem 3. Galton-Watson process with immigration. A *Galton-Watson process with immigration* is a discrete-time Markov chain $\{Z_n\}_{n \geq 0}$ used as a model for an asexually reproducing population in which new individuals from outside the population ("immigrants") are added to the population in each generation. The population evolves as follows:

(i) Each nth generation individual x produces a random number $\xi_x^{(n)}$ of $(n+1)$th generation offspring. The random variables $\xi_x^{(n)}$ for different individuals are independent, and have common offspring distribution

$$P\left\{\xi_x^{(n)} = k\right\} = p_k \qquad \text{for } k = 0, 1, 2, \ldots \qquad \text{where} \quad p_0 < 1,$$

with probability generating function

$$\varphi(s) = \sum_{k=0}^{\infty} p_k s^k.$$

(ii) In each generation $n \geq 1$ a random number ζ_n of individuals without parents in the $(n-1)$th generation ("immigrants") are added to the population. These individuals then reproduce in the $(n+1)$th generation according to the same law as the other individuals. The random variables $\zeta_1, \zeta_2, \ldots$ are mutually independent, and also independent of the random variables $\xi_x^{(n)}$, and have common distribution

$$P\{\zeta_n = k\} = q_k \qquad \text{for } k = 0, 1, 2, \ldots \qquad \text{satisfying } q_0 < 1,$$

with probability generating function

$$\beta(s) = \sum_{k=0}^{\infty} q_k s^k.$$

(iii) The state variable Z_n is the total number of individuals in the nth generation (including the nth generation immigrants).

(A) For all integers $1 \leq m \leq n$, define $Z_n^{(m)}$ to be the number of individuals in the nth generation who are descendants of immigrants who entered the population in the mth generation. Assume that $Z_0 = 1$. Find expressions for the probability generating functions

(1) $$H_n^{(m)}(s) := E s^{Z_n^{(m)}} \qquad \text{and}$$

(2) $$G_n(s) := E s^{Z_n}$$

in terms of the probability generating functions $\varphi(s)$ and $\beta(s)$ and their iterates $\varphi_n(s)$ and $\beta_n(s)$. HINT: You may find it useful to do (1) before (2).

(B) Assume that $\mu := \sum_k k p_k < 1$ and $\lambda := \sum_k k q_k < \infty$. Show that there is a stationary probability distribution $\{\pi_k\}_{k \geq 0}$ for the Markov chain Z_n.

HINT: It is enough to show that if $Z_0 = 0$ then the random variables Z_n converge in distribution as $n \to \infty$. For this, you may find the random variables $Z_n^{(m)}$ useful. What is $E Z_n^{(m)}$?

(C)* An exercise in the textbook *A First Course in Stochastic Processes* by KARLIN & TAYLOR asks the reader to show that there is a stationary probability distribution $\{\pi_k\}_{k \geq 0}$ for the Galton-Watson process with immigration Z_n whenever $\mu := \sum_k k p_k < 1$. This is *false*: In fact, if the immigration distribution $\left\{q_k\right\}_{k \geq 0}$ has *infinite mean*, and if the offspring distribution satisfies $p_0 + p_1 = 1$ and $0 < p_0 < 1$, then the Markov chain Z_n is *transient*. Prove this.

Problem 4. There's a Galton-Watson process in my random walk! Let S_n be the simple nearest-neighbor random walk on the integers started at $S_0 = 1$, and define T to be the first time $n \geq 1$ such that $S_n = 0$. Define $Z_0 = 1$ and for $k = 1, 2, 3, \ldots$ define

$$Z_k = \sum_{n=0}^{T-1} \mathbf{1}\{S_n = k \text{ and } S_{n+1} = k+1\},$$

that is, Z_k is the number of times that the random walk S_n crosses from k to $k+1$ before first visiting 0.

(A) Prove that the sequence $\{Z_k\}_{k \geq 0}$ is a Galton-Watson process, and identify the offspring distribution as a geometric distribution.

(B) Deduce the distribution of Z_k. Can you find an alternative explanation of your answer (perhaps using what you know about the gambler's ruin problem)?

(C) Show that $T = \sum_{k \geq 1} Z_k$ is the total number of individuals ever born in the course of the Galton-Watson process, and show that τ (the extinction time of the Galton-Watson process) is the maximum displacement M from 0 attained by the random walk before its first return to the origin.