Problem 1. Let $S_n$ be simple random walk on $\mathbb{Z}$ with initial point $S_0 = x$ under the probability measure $P^x$. Define $T = T_{-A,B}$ to be the first time $n \geq 0$ such that $S_n = -A$ or $S_n = +B$. Set

$$u(x) = E^x \sum_{j=1}^T \mathbf{1}_{\{S_n \geq 0\}}.$$ 

Write a difference equation for $u(x)$ and solve it.

Problem 2. Let $X_n$ be a positive recurrent, irreducible Markov chain on a finite or countable state space $\mathcal{X}$ with unique stationary distribution $\pi$, and for each state $y \in \mathcal{X}$ set $T_y = \min\{n \geq 1 : X_n = y\}$ and $N_{x,y} = \#\{n \leq T_x : X_n = y\}$.

(A) Show that $E^x N_{x,y} = \pi(y)/\pi(x)$.

(B) What is the distribution of $N_{x,y}$ under $P^x$? HINT: It should involve the parameters $\alpha = P^x\{T_y < T_x\}$ and $\beta = P^y\{T_y < T_x\}$.

(C) What is the relation between $\alpha$ and $\beta$?

Problem 3. Functions of Markov chains. Let $X_n$ be a Markov chain with transition probability matrix $P = (p(i,j))$ on the state space $\mathcal{X}$, and let $F : \mathcal{X} \to \mathcal{W}$ be onto, but not necessarily one-to-one. Set $W_n = F(X_n)$. When is $W_n$ a Markov chain?

(A) Show by example that $W_n$ might not be a Markov chain. HINT: You should be able to build an example where $\mathcal{X}$ has three states, $\mathcal{W}$ has two states, and the transition probabilities $p(i,j)$ are all either 0 or 1.

(B) It is sometimes the case that a function of a Markov chain is a Markov chain. Here is a sufficient condition: For each $w \in \mathcal{W}$, let $F^{-1}(w) = \{x \in \mathcal{X} : F(x) = w\}$. Suppose that for every pair $w_1, w_2$ of states in $\mathcal{W}$ and every pair $x_1, x_2 \in F^{-1}(w_1)$

$$\sum_{y \in F^{-1}(w_2)} p(x_1, y) = \sum_{y \in F^{-1}(w_2)} p(x_2, y).$$

Then $W_n$ is a Markov chain on $\mathcal{W}$. Prove this. Write a formula for the transition probabilities $q(w, w')$ of $W_n$. 

(C) Suppose that $W_n$ is a Markov chain. Suppose further that the covering Markov chain $X_n$ has a stationary distribution $\pi$. Show that the projection $\pi \circ F^{-1}$ of $\pi$ to $\mathcal{Y}$ is a stationary distribution for the Markov chain $W_n$.

**HINT:** Instead of driving yourself crazy trying to verify the equations in the definition of a stationary distribution, first prove the following:

**Lemma 1.** Let $P$ be a transition probability matrix on a state space $\mathcal{Y}$. Then a probability distribution $\mu$ on $\mathcal{Y}$ is stationary for $P$ if, on some probability space, there exist $\mathcal{Y}$-valued random variables $Y_0, Y_1$ such that (a) each of $Y_0$ and $Y_1$ has marginal distribution $\mu$; and (b) the conditional distribution of $Y_1$ given $Y_0 = y$ is the $y$th row of $P$.

(D) Here is an example: Let $X_n$ be the Ehrenfest urn model, as defined in the Lecture Notes. The state
\[ X_n = (X_n(1), X_n(2), \ldots, X_n(N)) \]
at any time $n$ is the vector of 0s and 1s indicating the locations of the $N$ balls. Define
\[ W_n := \sum_{j=1}^{N} X_n(j); \]
thus, $W_n$ is the number of balls in urn 1. Show that $W_n$ is a Markov chain, and find its stationary distribution. **NOTE:** In some of the literature, it is the process $W_n$ that is called the Ehrenfest model. This process is a discrete form of the *Ornstein-Uhlenbeck* process (also known in finance as the *Vasicek* model).

**Problem 4.** *Renewal Processes.* Let $A_n$ and $R_n$ be the age and residual lifetime at time $n$ in a renewal process with interoccurrence times $\xi_1, \xi_2, \ldots$. Assume that the interoccurrence times are i.i.d., with common distribution $\{q_m\}_{m \geq 1}$. Assume for simplicity that $q_m > 0$ for every $m \geq 1$.

(A) Explain (briefly) why the vector process $W_n := (A_n, R_n)$ is a Markov chain, and write a formula for its transition probabilities. **NOTE:** By convention, if there is an occurrence of the renewal process at time $n$ then $A_n = 0$ and $R_n =$ lifetime of the new component installed at time $n$. Thus, the state space of the process $W_n$ is $(\mathbb{N} \cup \{0\}) \times \mathbb{N}$, where $\mathbb{N} = \{1, 2, 3, \ldots\}$.

(B) Show that the Markov chain $W_n$ has a stationary distribution if and only if $\mu = \sum_{m=1}^{\infty} m q_m < \infty$. Show that if $\mu < \infty$, then the stationary distribution $\pi$ is unique, and find it.

(C) Use the results of Problem 3 to find the stationary distributions of the component chains $R_n$ and $A_n$ when $\mu < \infty$.

**Problem 5.** An $I \times J$ *contingency table* is a list $\{x_{i,j}\}_{i \leq I, j \leq J}$ of nonnegative integers indexed by ordered pairs $(i, j)$ with $1 \leq i \leq I$ and $1 \leq j \leq J$. The row and column totals are
\[ x_{i+} := \sum_{j=1}^{J} x_{i,j} \quad \text{and} \quad x_{+j} := \sum_{i=1}^{I} x_{i,j}. \]
Fix a set of row and column totals $x_{i+}$ and $x_{+j}$, and let $\mathcal{T}$ be the set of all contingency tables with these row and column totals. For two tables $x, y \in \mathcal{T}$, say that $x$ and $y$ are nearest neighbors if there exist pairs $i, i'$ and $j, j'$ such that

$$y_{ij} = x_{ij} \pm 1$$
$$y_{i'j} = x_{i'j} \mp 1$$

Prove that for any two tables $x, y$ there is a finite sequence of tables $x = t_0, t_1, \ldots, t_m = y$ such that any two successive tables $t_i, t_{i+1}$ are nearest neighbors. (This shows that the Markov chain described in class is irreducible.)