STATISTICS 312: STOCHASTIC PROCESSES
HOMEWORK ASSIGNMENT 5
DUE WEDNESDAY NOVEMBER 2

Problem 1. Symmetries. Let $\mathbb{P} = (p(i,j))$ be an irreducible transition probability matrix on a finite state space $\mathscr{Y}$. An *automorphism* (or *symmetry*) of the transition kernel $\mathbb{P}$ (or, more informally, of the the Markov chain with this transition kernel) is a one-to-one mapping $T : \mathscr{Y} \to \mathscr{Y}$ such that for every pair $i, j \in \mathscr{Y}$,

$$p(i,j) = p(T(i), T(j)).$$

Let π be the unique stationary probability distribution for the transition probability matrix $\mathbb{P}$. (Recall that the stationary distribution is unique if $\mathbb{P}$ is irreducible.) Suppose that $T : \mathscr{Y} \to \mathscr{Y}$ is an automorphism of $\mathbb{P}$.

(A) Show that for every $i \in \mathbb{Y}$,

$$\pi(i) = \pi(T(i)).$$

(B) Conclude that if for every pair i, j of states there is a symmetry π such that $\pi(i) = j$ then the stationary distribution must be the uniform distribution on $\mathscr{Y}$.

Problem 2. Top-to-random shuffling. Consider a deck of M cards, labeled $1, 2, 3, \dots, M$. In top-to-random shuffling, at each step n the top card of the deck is removed and then inserted at a random position in the deck. For example: if the current state of the deck is $(3, 4, 1, 2)$ then at the next step the possible states are

$$(3,4,1,2),$$
$$(4,3,1,2),$$
$$(4,1,3,2),$$
$$(4,1,2,3);$$

each of these has probability $\frac{1}{4}$. At each time $n = 0, 1, 2, \dots$ the state of the system is one of the $M!$ permutations of the integers $1, 2, 3, \dots, M$, so the state space is the set $\mathscr{S}_M$ of all permutations.

(A) Show that this Markov chain is irreducible and aperiodic.
(B) Show that the uniform distribution on $\mathscr{S}_M$ is a stationary distribution.

Problem 3. Reversibility. A Markov chain on the state space $\mathscr{X}$ with transition probabilities $p(x, y)$ is said to be *reversible* if there is a positive function $w : \mathscr{X} \to (0, \infty)$ such that for any two states $x, y \in \mathscr{X}$,

$$w(x)p(x,y) = w(y)p(y,x)$$

These equations are called the *detailed balance conditions.*

(A) Show that if the Markov chain is *irreducible* then the weight function w is unique up to multiplication by a scalar. HINT: First show that if the detailed balance equations hold, then for any $n \geq 1$ and $x, y \in \mathscr{X}$,

$$w(x)p_n(x, y) = w(y)p_n(y, x).$$

(B) Show that if the detailed balance equations hold for a weight function w such that $\sum_{x\in\mathscr{X}} w(x) = 1$ then w is a stationary distribution for the Markov chain.

(C) Show that an irreducible Markov chain on a finite or countable state space $\mathscr{X}$ is reversible if and only if for every finite *cycle* of states $x_0, x_1, x_2, \ldots, x_n = x_0$,

$$\prod_{i=0}^{n-1} p(x_i, x_{i+1}) = \prod_{i=0}^{n-1} p(x_{i+1}, x_i).$$

(D) Consider the p, q random walk on the discrete circle $\mathbb{Z}_m$ (i.e., the Markov chain that moves one step clockwise with probability p and one step counter-clockwise with probability $q = 1-p$). Is this Markov chain reversible?

Problem 4. Coupling and Total Variation. Let μ and ν be two probability distributions on a finite set $\mathscr{X}$. A *coupling* of μ and ν is a probability distribution λ on the Cartesian product $\mathscr{X} \times \mathscr{X}$ whose *marginal distributions* are μ and ν, that is,

$$\mu(x) = \sum_{y\in\mathscr{X}} \lambda(x, y) \quad \text{and}$$
$$\nu(y) = \sum_{x\in\mathscr{X}} \lambda(x, y).$$

Define a *maximal coupling* to be a coupling that assigns the largest possible probability to the *diagonal*

$$(\mathscr{X} \times \mathscr{X})_{\text{diagonal}} := \{(x, y) \in \mathscr{X} \times \mathscr{X} : x = y\}.$$

Prove that for any pair μ, ν of probability distributions on $\mathscr{X}$ there is a maximal coupling λ, and

$$\lambda(\mathscr{X} \times \mathscr{X}) - \lambda(\mathscr{X} \times \mathscr{X})_{\text{diagonal}} = \|\mu - \nu\|_{TV}.$$

Problem 5. A Queueing Model: This is a discrete-time Markov chain designed to model a queueing system with a single server. During each time period, several job requests are made of the server. The server can complete just one job in a single time period, so excess requests must be held in a queue to await processing during a later time period. Assume that the numbers of requests $Y_1, Y_2, Y_3, \ldots$ during time periods $1, 2, 3, \ldots$ are independent, identically distributed random variables with common distribution

$$P\{Y_i = k\} = a_k \qquad \text{for } k = 0, 1, 2, \ldots$$

Assume that $a_0 > 0$ that $a_0 < 1$. The state of the system at time n is just the number X_n of requests in the queue at the end of the nth time period. Thus, the transition probability matrix is:

$$\mathbb{P} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_0 & a_1 & \cdots \\ \cdots & & & & \end{pmatrix}$$

Observe that the first row breaks the pattern of the rest of the matrix. The reason is that, whenever the current state of the system is 0, the server is idle, and completes no jobs during the next time period; but whenever the current state is ≥ 1, the server will complete 1 job in the next time period.

Assume that the mean number $\mu = \sum_{k=0}^{\infty} k a_k$ of new requests per time period is < 1. The following 3 exercises will show that under this assumption the Markov chain has a stationary probability distribution $\pi_m = \pi(m)$.

(A) Let $\pi_m = \pi(m)$ be the stationary distribution of the Markov chain, and define the generating functions

$$G(z) = \sum_{m=0}^{\infty} \pi_m z^m \qquad \text{and} \qquad A(z) = \sum_{m=0}^{\infty} a_m z^m.$$

Derive a functional equation relating $G(z)$ and $A(z)$. HINT: Begin writing the defining equations for a stationary distribution for $\pi_0, \pi_1, \pi_{2,\dots}$. Multiply these by $z^0, z^1, z^2, \dots$, sum, and simplify. When I tried this myself, I obtained (I think)

$$G(z) = A(z)\{\pi_0 + (G(z) - \pi_0)/z\}. \tag{1}$$

(B) Consider the special case where $a_m = qp^m$ for some $0 < p < 1$ and $q = 1 - p$. Use the results of (c) and (d) to give exact formulas for the steady-state probabilities π_m.

(C) Show that, when $\mu < 1$ and with $\pi_0 = (1 - \mu)$, the equation (**??**), when solved for $G(z)$, defines a function that is a probability generating function. The functional equation then implies (why?) that the coefficients π_m of $G(z)$ define a stationary distribution.