Problem 1. Let \( \{ N_j \} \) be a Poisson point process in \( \mathbb{R}^2 \) with intensity function \( \lambda(x,y) = h(y) \) that depends only on the \( y \)-coordinate. Let \( N_R \) be the number of points in the rectangle \( R = [0, T] \times [0, A] \), and denote by \((X_i, Y_i)\), where \( i = 1, 2, \ldots, N_R \), the locations of the occurrences in the region \( R \).

(A) Show that the conditional distribution of the values \( Y_1, Y_2, \ldots, Y_n \) given the event \( N_R = n \) is the same as the distribution of \( n \) points sampled independently from the density

\[
f_a(y) := \frac{h(y)}{\iint_R h(y) \, dx \, dy}.
\]

(B) Show that the characteristic function of \( \sum_{i=1}^{N_R} Y_i \) is

\[
E \exp\{i \theta \sum_{i=1}^{N_R} Y_i\} = \exp\{-\alpha_R + \alpha_R \varphi(\theta)\}
\]

where

\[
\alpha_R = \iint_R h(y) \, dx \, dy \quad \text{and} \quad \varphi(\theta) = \int_0^A e^{i \theta y} f_a(y) \, dy.
\]

Problem 2. Asymmetric Random Walk. This problem is concerned with the \( p - q \) random walk on the integers, that is, the nearest neighbor random walk in which jumps to the right occur with probability \( p \) and jumps to the left with probability \( q = 1 - p \). Let \( S_n \) be the position after \( n \) steps; then

\[
S_n = x + \sum_{i=1}^{n} \xi_i
\]

where \( \xi_1, \xi_2, \ldots \) are i.i.d. Rademacher-\( p \), that is,

\[
P\{\xi_j = +1\} = p, \quad P\{\xi_j = -1\} = q,
\]

Dependence of probabilities and expectations on the initial state \( x \) will be indicated by putting a superscript \( x \) on the probability and expectation operators \( P \) and \( E \). Fix a positive integer \( M \).
and an arbitrary integer $a$ and let
\begin{equation}
\tau = \tau_{[0,M]} = \inf\{n : S_n = 0 \text{ or } M\}
\end{equation}
\begin{equation}
T_a = \inf\{n : S_n = a\}.
\end{equation}

(A) Write and solve a difference equation for $u(x) := P^x \{S_\tau = M\}$.
(B) Write and solve a difference equation for the expected time of exit $v(x) = E^x \tau$.

**Problem 3. Ballot Problem.** An election is held with two candidates $A$ and $B$. A total of $N$ voters cast ballots; candidate $A$ receives $N_A$ votes and candidate $B$ receives $N_B = N - N_A$ votes. Assume that $N_A \geq N_B$. Suppose the votes are drawn from the ballot box in random order and the votes are tallied one at a time. What is the probability that at any stage of the tally candidate $B$ is ahead (by at least one vote) in the count? Solve this using a reflection argument, as follows:

(A) What is the total number $\mathcal{E}(N_A, N_B)$ of random orderings of the ballots, subject to the constraint that $N_A$ are labeled $A$ and $N_B = N - N_A$ are labeled $B$?

(B) Let $S_n^A$ and $S_n^B$ be the tallies for $A$ and $B$ after $n$ ballots have been observed. Consider the orderings in which at some stage candidate $B$ pulls ahead. There must be a first time $\tau \geq 1$ at which $S_n^B - S_n^A = 1$. Suppose that at this time all of the remaining ballots are flipped, that is, ballots for $A$ are relabeled $B$ and ballots for $B$ are relabeled $A$. Show that the election would then result in (I think!) $N_A + 1$ votes for candidate $B$ and $N_B - 1$ for candidate $A$.

(C) Show that the relabeling procedure in part (B) sets up a one-to-one correspondence between (a) the ballot orderings in which at some stage candidate $B$ pulls ahead but ends up with only $N_A$ votes and (b) the ballot orderings in which candidate $B$ wins with $N_A + 1$ votes. **Hint:** The procedure in part (B) is reversible.

(D) Conclude that the number of ballot orderings in which candidate $A$ receives $N_A$ votes, candidate $B$ receives $N_B$ votes, and candidate $B$ never leads in the count, is equal to
\begin{equation}
\mathcal{E}(N_A, N_B) - \mathcal{E}(N_A + 2, N_B - 2).
\end{equation}
Use this and the result of part (A) to calculate the probability that at some stage of the tally candidate $B$ is ahead (by at least one vote) in the count.

(E) **Generalization:** Assume as in the original problem that candidate $A$ receives $N_A$ votes and candidate $B$ receives $N_B = N - N_A$ votes, and assume that $N_B \geq k$ for some $k \geq 1$. What is the probability that at any stage of the tally candidate $B$ is ahead in the vote count by at least $k$ votes?

**Problem 4. Ballot Problem and Random Walk.** Let $S_n$ be a $p-q$ random walk starting at $S_0 = 0$, that is, the steps $X_j = S_j - S_{j-1}$ are independent, identically distributed with distribution
\begin{align*}
P_p\{X_j = +1\} &= p, \\
P_p\{X_j = -1\} &= q,
\end{align*}
where $p + q = 1$. 

(A) Show that conditional on the event $S_N = 2m$, all possible orderings of the steps $X_1, X_2, \ldots X_N$ are equally likely, that is, their conditional distribution is the same as in sampling without replacement from an urn with $N + m$ ballots marked +1 and $N - m$ ballots marked −1.

(B) Use the result of the ballot problem above to give a formula for the first-passage probability

$$P_p\{T = 2N + 1\}$$

where $T$ is the first time (if ever) that the random walk $S_n$ reaches +1, that is,

$$T = \min\{n : S_n = +1\}.$$