STATISTICS 312: STOCHASTIC PROCESSES
HOMEWORK ASSIGNMENT 2
DUE MONDAY OCTOBER 10

Problem 1. Let $\{N_J\}$ be a Poisson point process in $\mathbb{R}^2$ with intensity function $\lambda(x,y) = h(y)$ that depends only on the $y-$coordinate. Let N_R be the number of points in the rectangle $R = [0,T] \times [0,A]$, and denote by (X_i, Y_i), where $i = 1,2,\dots,N_R$, the locations of the occurrences in the region R.

(A) Show that the conditional distribution of the values $Y_1, Y_2, \dots, Y_n$ given the event $N_R = n$ is the same as the distribution of n points sampled independently from the density

$$f_A(y) := \frac{h(y)}{\iint_R h(y)\,dx\,dy}.$$

(B) Show that the *characteristic function* of $\sum_{i=1}^{N_R} Y_i$ is

$$E\exp\{i\theta\sum_{i=1}^{N_R} Y_i\} = \exp\{-\alpha_R + \alpha_R\varphi(\theta)\}$$

where

$$\alpha_R = \iint_R h(y)\,dx\,dy \quad \text{and} \quad \varphi(\theta) = \int_0^A e^{i\theta y} f_A(y)\,dy.$$

Problem 2. Asymmetric Random Walk. This problem is concerned with the $p-q$ random walk on the integers, that is, the nearest neighbor random walk in which jumps to the right occur with probability p and jumps to the left with probability $q = 1-p$. Let S_n be the position after n steps; then

$$S_n = x + \sum_{i=1}^n \xi_i \tag{1}$$

where $\xi_1, \xi_2, \dots$ are i.i.d. Rademacher-p, that is,

$$P\{\xi_j = +1\} = p,$$
$$P\{\xi_j = -1\} = q,$$

Dependence of probabilities and expectations on the initial state x will be indicated by putting a superscript x on the probability and expectation operators P and E. Fix a positive integer M

and an arbitrary integer a and let

$$\tau = \tau_{[0,M]} = \inf\{n : S_n = 0 \text{ or } M\} \tag{2}$$

$$T_a = \inf\{n : S_n = a\}. \tag{3}$$

(A) Write and solve a difference equation for $u(x) := P^x\{S_\tau = M\}$.
(B) Write and solve a difference equation for the expected time of exit $v(x) = E^x\tau$.

Problem 3. Ballot Problem. An election is held with two candidates A and B. A total of N voters cast ballots; candidate A receives N_A votes and candidate B receives $N_B = N - N_A$ votes. Assume that $N_A \geq N_B$. Suppose the votes are drawn from the ballot box in random order and the votes are tallied one at a time. What is the probability that at any stage of the tally candidate B is ahead (by at least one vote) in the count? Solve this using a reflection argument, as follows:

(A) What is the total number $\mathscr{E}(N_A, N_B)$ of random orderings of the ballots, subject to the constraint that N_A are labeled A and $N_B = N - N_A$ are labeled B?

(B) Let S_n^A and S_n^B be the tallies for A and B after n ballots have been observed. Consider the orderings in which at some stage candidate B pulls ahead. There must be a first time $\tau \geq 1$ at which $S_\tau^B - S_\tau^A = 1$. Suppose that at this time all of the *remaining* ballots are *flipped*, that is, ballots for A are relabeled B and ballots for B are relabeled A. Show that the election would then result in (I think!) $N_A + 1$ votes for candidate B and $N_B - 1$ for candidate A.

(C) Show that the relabeling procedure in part (B) sets up a one-to-one correspondence between (a) the ballot orderings in which at some stage candidate B pulls ahead but ends up with only N_A votes and (b) the ballot orderings in which candidate B wins with $N_A + 1$ votes. HINT: The procedure in part (B) is reversible.

(D) Conclude that the number of ballot orderings in which candidate A receives N_A votes, candidate B receives N_B votes, and candidate B never leads in the count, is equal to

$$\mathscr{E}(N_A, N_B) - \mathscr{E}(N_A + 2, N_B - 2).$$

Use this and the result of part (A) to calculate the probability that at some stage of the tally candidate B is ahead (by at least one vote) in the count.

(E) *Generalization:* Assume as in the original problem that candidate A receives N_A votes and candidate B receives $N_B = N - N_A$ votes, and assume that $N_B \geq k$ for some $k \geq 1$. What is the probability that at any stage of the tally candidate B is ahead in the vote count by at least k votes?

Problem 4. Ballot Problem and Random Walk. Let S_n be a $p - q$ random walk starting at $S_0 = 0$, that is, the steps $X_j = S_j - S_{j-1}$ are independent, identically distributed with distribution

$$P_p\{X_j = +1\} = p,$$
$$P_p\{X_j = -1\} = q,$$

where $p + q = 1$.

(A) Show that *conditional* on the event $S_N = 2m$, all possible orderings of the steps $X_1, X_2, \dots X_N$ are equally likely, that is, their conditional distribution is the same as in sampling *without* replacement from an urn with $N+m$ ballots marked $+1$ and $N-m$ ballots marked -1.

(B) Use the result of the ballot problem above to give a formula for the first-passage probability

$$P_p\{T = 2N+1\}$$

where T is the first time (if ever) that the random walk S_n reaches $+1$, that is,

$$T = \min\{n : S_n = +1\}.$$