Problem 1. Let $N_t$ and $M_t$ be independent Poisson counting processes with intensities $\nu, \mu$, respectively. Define $\tau$ to be the time of first occurrence in the process $N_t$, so that $\tau = \min\{t : N_t = 1\}$.

(A) What is the distribution of $M_\tau$?
(B) What is the distribution of $N_{2\tau} - N_\tau$?

Problem 2. Record values. Let $U_1, U_2, \ldots$ be independent, identically distributed random variables with the uniform-$[0, 1]$ distribution. Define the record (low) value times $\sigma_n = \sigma(n)$ inductively as follows:

\[
\sigma_1 = 1; \\
\sigma_{n+1} = \min\{k : U_k < U_{\sigma(n)}\}.
\]

(A) Define $W_n = U_{\sigma(n)} / U_{\sigma(n-1)}$: this is the relative decrease in the record low at the time of the $n$th record value. Show that $W_n$ is uniformly distributed on $[0, 1]$, and that $W_n$ is independent of $\sigma_n$ and $U_{\sigma(n)-1}$.

(B) Conclude that the distribution of $U_{\sigma(n)}$ is the same as the distribution of the product $\prod_{i=1}^{n} U_i$.

(C) Now let $X_1, X_2, \ldots$ be independent, identically distributed random variables with the unit exponential distribution. Define the record high value times $\nu_n = \nu(n)$ by

\[
\nu_1 = 1; \\
\nu_{n+1} = \min\{k : X_k > X_{\nu(n)}\}.
\]

Show that the sequence of record values $X_{\sigma(1)}, X_{\sigma(2)}, \ldots$ is a Poisson point process of unit intensity.

Problem 3. Let $N(t)$ be a Poisson process with intensity $\lambda > 0$, and let $T_1 < T_2 < \cdots$ be the occurrence times. Let $T_0 = 0$. At any time $t > 0$, the time elapsed since the last occurrence is $A_t := t - T_{N(t)}$, and the time remaining until the next occurrence is $R_t := T_{N(t)+1} - t$. (The letters $A$ and $R$ are commonly used to mean age and residual lifetime.) We know (why?) that for any $t > 0$ the distribution of $R_t$ is exponential—$\lambda$.

(A) What is the distribution of $A_t$?
(B) Show that $A_t$ and $R_t$ are independent.
(C) Show that $A_t$ converges in distribution as $t \to \infty$.

(D) The random variable $A_t + R_t$ is the length of the inter-occurrence interval containing $t$. Use (A)–(C) to deduce that $A_t + R_t$ converges in distribution as $t \to \infty$, and observe that the limit distribution is not the exponential—$\lambda$ distribution. Explain the apparent paradox.
Problem 4. Particles enter a linear accelerator (think of this as the half-line $\mathbb{R}_+$) at location 0 at the occurrence times of a Poisson process $N(t)$ with intensity $\beta$. The $n$th particle has velocity $V_n$; the sequence $V_1, V_2, \ldots$ consists of independent, identically distributed random variables with distribution $F$, and these are independent of the Poisson process $N(t)$. Assume that when a faster particle passes a slower one there is no interaction between the two.

Assume that $F$ is the discrete uniform distribution on the set $[K] := \{1, 2, \ldots, K\}$. Let $M_t(x)$ be the number of particles located in the interval $[x, \infty)$ at time $t$. What is the distribution of $M_t(x)$?

Problem 5. Convergence to the Poisson distribution. The Poisson distribution is often a good approximation of the distribution of a random variable obtained by counting the number of a large number of rare events. This is true even when the events in question are dependent, provided the dependence is (in a suitable sense) weak. In many such situations, the method of moments is a useful tool for proving convergence to the Poisson distribution. This problem gives an example of how the method of moments is employed.

(A) Let $W_n$ be a sequence of nonnegative integer-valued random variables such that for each $k = 1, 2, \ldots,$

$$\lim_{n \to \infty} E\left(\frac{W_n}{k!}\right) = \frac{\lambda^k}{k!}.$$ 

Prove that $W_n$ converges in distribution to the Poisson distribution with mean $\lambda$. NOTE: By convention, if $r > m$ then $\binom{r}{m} = 0$.

In parts (B)–(C), $W_n$ is the number of fixed points of a random permutation of the integers $[n] := \{1, 2, \ldots, n\}$. You may think of $W_n$ as follows: Suppose that $n$ people check their coats at a coat check. At the end of the evening the coats are randomly re-distributed. The number of people who wind up with their own coats is $W_n$.

(B) Show that for any $k \geq 1$,

$$E\left(\frac{W_n}{k}\right) = \sum_{A \subset [n]:|A|=k} P\{\text{every individual in } A \text{ gets own coat back}\}.$$ 

(C) Conclude that as $n \to \infty$ the random variables $W_n$ converge in distribution to the Poisson distribution with mean 1.