STATISTICS 312: STOCHASTIC PROCESSES
HOMEWORK ASSIGNMENT 1
DUE MONDAY OCTOBER 3

Problem 1. Let N_t and M_t be independent Poisson counting processes with intensities ν, μ, respectively. Define τ to be the time of first occurrence in the process N_t, so that $\tau = \min\{t : N_t = 1\}$.

(A) What is the distribution of M_τ?
(B) What is the distribution of $N_{2\tau} - N_\tau$?

Problem 2. Let $N(t)$ be a Poisson process with intensity $\lambda > 0$, and let $T_1 < T_2 < \cdots$ be the occurrence times. Let $T_0 = 0$. At any time $t > 0$, the time elapsed since the last occurrence is $A_t := t - T_{N(t)}$, and the time remaining until the next occurrence is $R_t := T_{N(t)+1} - t$. (The letters A and R are commonly used to mean *age* and *residual lifetime*.) We know (why?) that for any $t > 0$ the distribution of R_t is exponential–λ.

(A) What is the distribution of A_t?
(B) Show that A_t and R_t are independent.
(C) Show that A_t converges in distribution as $t \to \infty$.

(D) The random variable $A_t + R_t$ is the length of the inter-occurrence interval containing t. Use (A)–(C) to deduce that $A_t + R_t$ converges in distribution as $t \to \infty$, and observe that the limit distribution is *not* the exponential–λ distribution. Explain the apparent paradox.

Problem 3. Particles enter a linear accelerator (think of this as the half-line $\mathbb{R}_+$) at location 0 at the occurrence times of a Poisson process $N(t)$ with intensity β. The nth particle has velocity V_n; the sequence $V_1, V_2, \ldots$ consists of independent, identically distributed random variables with distribution F, and these are independent of the Poisson process $N(t)$. Assume that when a faster particle passes a slower one there is no interaction between the two.

Assume that F is the discrete uniform distribution on the set $[K] := \{1, 2, \ldots, K\}$. Let $M_t(x)$ be the number of particles located in the interval $[x, \infty)$ at time t. What is the distribution of $M_t(x)$?

Problem 4. Convergence to the Poisson distribution. The Poisson distribution is often a good approximation of the distribution of a random variable obtained by counting the number of a large number of rare events. This is true even when the events in question are dependent, provided the dependence is (in a suitable sense) weak. In many such situations, the *method of moments* is a useful tool for proving convergence to the Poisson distribution. This problem gives an example of how the method of moments is employed.

(A) Let W_n be a sequence of nonnegative integer-valued random variables such that for each $k = 1, 2, \ldots$,

$$\lim_{n\to\infty} E\binom{W_n}{k} = \frac{\lambda^k}{k!}.$$

Prove that W_n converges in distribution to the Poisson distribution with mean λ. NOTE: By convention, if $r > m$ then $\binom{n}{r} = 0$.

In parts (B)– (C), W_n is the number of fixed points of a random permutation of the integers $[n] := \{1, 2, \dots, n\}$. You may think of W_n as follows: Suppose that n people check their coats at a coat check. At the end of the evening the coats are *randomly* re-distributed. The number of people who wind up with their own coats is W_n.

(B) Show that for any $k \geq 1$,

$$E\binom{W_n}{k} = \sum_{A \subset [n] : |A| = k} P\{\text{every individual in } A \text{ gets own coat back}\}.$$

(C) Conclude that as $n \to \infty$ the random variables W_n converge in distribution to the Poisson distribution with mean 1.

Problem 5. Record values. Let $U_1, U_2, \dots$ be independent, identically distributed random variables with the uniform-$[0,1]$ distribution. Define the *record (low) value times* $\sigma_n = \sigma(n)$ inductively as follows:

$$\sigma_1 = 1;$$
$$\sigma_{n+1} = \min\{k : U_k < U_{\sigma(n)}\}.$$

(A) Define $W_n = U_{\sigma(n)}/U_{\sigma(n-1)}$: this is the relative decrease in the record low at the time of the nth record value. Show that W_n is uniformly distributed on $[0,1]$, and that W_n is independent of σ_n and $U_{\sigma(n-1)}$.

(B) Conclude that the distribution of $U_{\sigma(n)}$ is the same as the distribution of the product $\prod_{i=1}^{n} U_i$.

(C) Now let $X_1, X_2, \dots$ be independent, identically distributed random variables with the unit exponential distribution. Define the record high value times $\nu_n = \nu(n)$ by

$$\nu_1 = 1;$$
$$\nu_{n+1} = \min\{k : X_k > X_{\nu(n)}\}.$$

Show that the sequence of record values $X_{\sigma(1)}, X_{\sigma(2)}, \dots$ is a Poisson point process of unit intensity.