STATISTICS 310: OPTIMIZATION & SIMULATION
PROBLEM SET 1: PERCOLATION

These exercises deal with Bernoulli-p site percolation on a connected, infinite graph $G = (\mathscr{V}, \mathscr{E})$. The setup is this: To each vertex $v \in \mathscr{V}$ is attached a Bernoulli-p random variable X_v, in such a way that the different random variables X_v are mutually independent. A vertex v is said to be *blue*(or *open*) if $X_v = 1$; it is *red* (or *closed*) if $X_v = 0$. Each vertex v is part of a maximal connected cluster K_v of blue vertices (by convention, $K_v = \emptyset$ if v is red). Say that *percolation* occurs if there is a vertex v such that $|K_v| = \infty$.

Problem 1. Let $G = (V, \mathscr{E})$ be an infinite, connected graph with *bounded degree*, that is, assume that there exists an integer $2 \le d < \infty$ such that no vertex $v \in V$ has more than d neighbors. Consider Bernoulli site percolation on G.

(a) Prove that if $p < 1/(d-1)$ then $E|K_v| < \infty$ for every vertex v.
(b) Prove that if $p = 1/(d-1)$ then $P\{|K_v| = \infty\} = 0$.

HINT: Think about growing the cluster K_v recursively.

Problem 2. *Peierls' Contour Argument.* Assume now that G is the usual integer lattice $\mathbb{Z}^2$ in dimension 2. (Thus, every vertex $(x, y) \in \mathbb{Z}^2$ has as neighbors the four vertices $(x \pm 1, y \pm 1)$.) Let G^* be the *augmented lattice*: this has the same vertex set $\mathbb{Z}^2$, but each vertex $v = (x, y)$ now has 8 neighbors

$$\begin{array}{ll} (x, y+1) & (x+1, y+1) \\ (x, y-1) & (x+1, y-1) \\ (x+1, y) & (x-1, y+1) \\ (x-1, y) & (x-1, y-1). \end{array}$$

(a) Prove that any *finite* blue cluster K_v must be surrounded by a closed red path in the augmented graph G^*. If $K_{(0,0)}$ has cardinality m, then the surrounding path must have length at least $\sqrt{m}/4$ (or maybe $\sqrt{m}/8$ or $\sqrt{m}/16$) distant from $(0, 0)$.

(b) Show that the number of closed paths of length $L \ge 4$ that enclose the origin is no larger than $32L \times 7^{L-1}$ (or something like that).

(c) Now use the Borel-Cantelli Lemma (easy half) to show that if $1 - p < 1/7$ then there is positive probability that there is *no* closed red path in the augmented graph G^* that surrounds $(0, 0)$. Hence, there is positive probability that $K_{(0,0)}$ is infinite.

Problems 1–2 show that for the standard two-dimensional lattice the critical probability p_c for site percolation is between 1/3 and 6/7. We would like to narrow that range down a bit. For this we'll resort to Monte Carlo.

Problem 3. (a) Write a MATLAB function that will take as input a rectangular array $(x_{i,j})$ of 0s and 1s and a vertex (i,j) and output the connected cluster $K_{(i,j)}$ of vertices marked 1 that contain (i,j). You can base your program on either breadth-first or depth-first search.

(b) Use your function on square arrays of Bernoulli-p random variables of moderate[1] size $(101 \times 101?)$ for various values of p to estimate the probability that the connected cluster K_v containing the point at the center of your array reaches one of the edges of the square. This should give you a rough estimate of p_c.

(c) *Optional* Write a MATLAB program that will grow percolation clusters from $(0,0)$ *dynamically*, that is, doen't first fill an entire $M \times M$ array of random Bernoullis. Your program should maintain a list of vertices that need to be tested, and a separate *dictionary* of vertices that have been added to the cluster. In MATLAB you can use the `Map` class to implement a dictionary (I think). If you're successful, you should be able to grow much larger clusters than by using the kind of program you wrote for (a), and thereby get much more precise estimates for p_c.

[1] First try much smaller arrays to make sure that your program won't freeze the department computers for the next 15 years!