Problem 1. Suppose that a particle is fired from the origin in the \((x, y)\)-plane in a straight line in a direction at a random angle \(\Theta\) to the \(x\)-axis. Assume that \(\Theta\) has the uniform distribution on the interval \((-\pi/2, \pi/2)\), so that the particle will eventually cross the line \(x = 1\). Let \((1, Y)\) be the (random) point at which the particle crosses the line \(x = 1\). Show that \(Y\) has the Cauchy density
\[
f_Y(y) = \frac{1}{\pi(1 + y^2)}.
\]

Note: See Pitman, page 310 for a diagram.

Problem 2. (a) Let \(X, Y\) be two independent random variables, each with the uniform distribution on \([0, 1]\). Find the p. d. f. of \(X + Y\).
(b) Let \(X, Y\) be two independent random variables, both exponentially distributed with mean 1. Find the p. d. f. of \(Y/X\).

Problem 3. Let \((X, Y)\) have joint density \(f(x, y) = ce^{-x^2-y^2-xy-x}\) where \(c > 0\) is a numerical constant. Compute the marginal density of \(X\).

Problem 4. Let \(U_1, U_2, \ldots\) be independent random variables all uniformly distributed on the unit interval. Show that for any positive number \(t\),
\[
\lim_{n \to \infty} P\{\min_{1 \leq i \leq n} U_i \leq t/n\}
\]
evaluates to \(\log(n+1)/n\), and identify the limit.

Problem 5. Let \(U_1, U_2, \ldots\) be independent random variables all uniformly distributed on the unit interval, and let \(N\) be the first integer \(n \geq 2\) such that \(U_n > U_{n-1}\). Show that for each real number \(0 \leq u \leq 1\)
(a) \(P(U_1 \leq u \text{ and } N = n) = \frac{n^{n-1}}{(n-1)!} - \frac{n^n}{n!}\).
(b) \(P(U_1 \leq u \text{ and } N \text{ is even}) = 1 - e^{-u}\).
(c) \(E N = e\).

Problem 6. Let \(m, n \geq 1\) be two positive integers, and let \(X_0, X_1, X_2, \ldots, X_{m+n}\) be independent random variables all with the uniform density on the unit interval \([0, 1]\). Let \(A_m\) be the event that \(X_0\) is the \((m + 1)\)th largest number in the sample \(X_0, X_1, X_2, \ldots, X_{m+n}\). (In other words, \(A_m\) is the event that there are exactly \(m\) values in the sample \(X_1, X_2, \ldots, X_{m+n}\) such that \(X_i \leq X_0\).)
(a) Without evaluating any integrals, evaluate \(P(A_m)\).
(b) Without using the answer to (a), show that
\[
P(A_m) = \int_0^1 \cdots \int_{F_m} 1 \, dx_1 dx_2 \ldots dx_{m+n} dx_0
\]
where \( F_m \) is the set of all points \((x_0, x_1, \ldots, x_{m+n})\) such that exactly \( m \) of the coordinates \( x_1, x_2, \ldots, x_{m+n} \) are below \( x_0 \).

(c) Show that for each value of \( x_0 \), the inner \( (m+n) \)-fold integral in part (b) evaluates to
\[
\binom{n+m}{m} x_0^m (1-x_0)^n.
\]

(d) Use the results of (a), (b), (c) to evaluate the integral
\[
\int_0^1 v^m (1-v)^n \, dv.
\]