STATISTICS 251
HOMEWORK ASSIGNMENT 6
DUE FRIDAY NOVEMBER 3

Problem 1. Suppose that a particle is fired from the origin in the $(x, y)-$plane in a straight line in a direction at a random angle Θ to the $x-$axis. Assume that Θ has the uniform distribution on the interval $(-\pi/2, \pi/2)$, so that the particle will eventually cross the line $x = 1$. Let $(1, Y)$ be the (random) point at which the particle crosses the line $x = 1$. NOTE: See Pitman, page 310 for a diagram.

(a) Show that Y has the *Cauchy* density

$$f_Y(y) = \frac{1}{\pi(1 + y^2)}.$$

(b) Suppose now that Θ has the uniform distribution on the interval $(0, \pi/2)$, so that the point $(1, Y)$ of intersection has a *positive* $y-$coordinate Y. What is EY?

Problem 2. Let $U_1, U_2, \dots$ be independent random variables all uniformly distributed on the unit interval.

(a) Show that for any positive number t,

$$\lim_{n\to\infty} P\{\min_{1\leq i\leq n} U_i \leq t/n\}$$

exists, and identify the limit. HINT: $x = e^{\log x}$.

(b) Assume that n is odd, and let M_n be the *sample median* of the random variables $U_1, U_2, \dots, U_n$. Show that for any real number t,

$$\lim_{n\to\infty} P\{\sqrt{n}(M_n - \frac{1}{2}) \leq t\}$$

exists, and find it.

Problem 3. Buses arrive at 116th and Broadway at the times of a Poisson arrival process with intensity λ arrivals per hour. These may either be M104 buses or M6 buses; the chance that a bus is an M104 is .6, while the chance that it is an M6 is .4, and the types (M6 or M104) of successive buses are independent.

(a) If I wait for an M104 bus, what is the chance that I will wait longer than x hours?
(b) What is the probability that two M6 buses and no M104 buses arrive in the first x hours?
(c) What is the expected number of hours until the third M6 arrives?
(d) What is the *variance* of the number of hours until the third M6 arrives?

Problem 4. Let $U_1, U_2, \ldots$ be independent random variables all uniformly distributed on the unit interval, and let N be the first integer $n \geq 2$ such that $U_n > U_{n-1}$. Show that for each real number $0 \leq u \leq 1$,

(a) $P(U_1 \leq u \text{ and } N = n) = \frac{u^{n-1}}{(n-1)!} - \frac{u^n}{n!}$.

(b) $P(U_1 \leq u \text{ and } N \text{ is even}) = 1 - e^{-u}$.

(c) $EN = e$.

Problem 5. Let $m, n \geq 1$ be two positive integers, and let $X_0, X_1, X_2, \ldots, X_{m+n}$ be independent random variables all with the uniform density on the unit interval $[0, 1]$. Let A_m be the event that X_0 is the $(m+1)$th largest number in the sample $X_0, X_1, X_2, \ldots, X_{m+n}$. (In other words, A_m is the event that there are exactly m values in the sample $X_1, X_2, \ldots, X_{m+n}$ such that $X_i \leq X_0$.)

(a) Without evaluating any integrals, evaluate $P(A_m)$.

(b) Without using the answer to (a), show that

$$P(A_m) = \iint \ldots \int_{F_m} 1\, dx_1 dx_2 \ldots dx_{m+n} dx_0$$

where F_m is the set of all points $(x_0, x_1, \ldots, x_{m+n})$ such that exactly m of the coordinates $x_1, x_2, \ldots x_{m+n}$ are below x_0.

(c) Show that for each value of x_0, the inner $(m+n)-$fold integral in part (b) evaluates to

$$\binom{n+m}{m} x_0^m (1 - x_0)^n.$$

(d) Use the results of (a), (b), (c) to evaluate the integral

$$\int_0^1 v^m (1 - v)^n \, dv.$$