**STATISTICS 251**
**HOMEWORK ASSIGNMENT 4**
**DUE FRIDAY OCTOBER 20**

**Problem 1.** Let $X$ and $Y$ be two independent geometric random variables, both with parameter $p$; thus, for each $k = 1, 2, 3, \ldots$

$$P(X = k) = P(Y = k) = p(1-p)^{k-1}.$$

Define $U = \min(X, Y)$ and $V = \max(X, Y)$. Find the joint distribution of $U$ and $V$.

**Problem 2.** Let $X$ and $Y$ be two independent Poisson random variables, with parameters $\lambda$ and $\mu$, respectively; thus, for each $k = 0, 1, 2, 3, \ldots$

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!} \quad \text{and}$$
$$P(Y = k) = \frac{\mu^k e^{-\mu}}{k!}.$$

Define $Z = X + Y$. Find the *conditional* distribution of $X$ given $Z$, that is for any $0 \le k \le n$ find

$$P(X = k \mid Z = n).$$

NOTE: Simplify your answer to the extent possible. You should recognize the answer when you've got it. HINT: You will need a formula for $P(Z = n)$. This you can find in section 3.5 of Pitman.

**Problem 3.** The *probability generating function* of a nonnegative, integer-valued random variable $X$ is the function $\varphi_X(t)$ defined by the power series

$$\varphi_X(t) = \sum_{n=0}^{\infty} t^n P\{X = n\}.$$

(A) What is the probability generating function of a random variable $S$ with the Binomial-$(n, p)$ distribution? of a random variable $N$ with the Poisson-$\lambda$ distribution?

(B) Let $X$ and $Y$ be independent, nonnegative, integer-valued random variables. What is the probability generating function of $X + Y$?

(C) Use your answers to (A) and (B) to show that if $X$ and $Y$ are independent random variables with distributions $X \sim$Poisson-$\lambda$ and $Y \sim$Poisson-$\mu$ then $X + Y$ is Poisson with parameter $\lambda + \mu$.

**Problem 4.** Recall the *hat-check* problem: $n$ people give their hats to a hat-check attendant, who then returns them randomly, one to each person. (Thus, all possible re-assignments of hats to people are equally likely.) Let $X$ be the number of people who receive their own hats back. Calculate $\mathrm{Var}X$.

**Problem 5.** Each box of Cap'n Crunch cereal contains an action figure, e.g., the Arnold Schwarzenegger Terminator action figure, the Captain America action figure, the T. S. Eliot action figure, etc.). There are $k$ different action figures, each of which occurs with frequencies $1/k$; thus, each time you buy a new box of CC the probability that it contains action figure $i$ is $1/k$, independent of the action figures you obtained in earlier purchases. Let $T_k$ be the total number of boxes you have bought by the time you get your $k$th different action figure.

(a) What is the distribution of $T_2 - T_1$?
(b) What is the expectation $ET_k$ of the number of boxes of CC that you will need to buy to obtain a complete set of action figures?
(c) What is the *conditional* distribution of $T_3 - T_2$ *given* that $T_2 = 13$?

**Problem 6.** In anticipation of the new Cap'n Crunch action figure promotion, the Treasure Island grocery store must decide how many boxes $b$ to stock. The demand $Y$ (the number of people who will want to buy a box) is unknown, so the manager treats it as a random variable with distribution $P\{Y = k\}$, with $k = 0, 1, 2, \dots$. Each sale of a box of CC will bring in \$ $A$ in profit, while each unsold box in stock will result in a loss of \$ $B$, so if the total demand turns out to be $Y = k$ then the net profit will be

$$W = \begin{cases} Ak - B(b-k) & \text{if } k \leq b \\ Ab & \text{if } k > b. \end{cases}$$

Treasure Island wants to *maximize* its expected profit $EW$. Show that the value of $b$ that will maximize $EW$ is the least integer $b_*$ such that

$$P\{Y \leq b_*\} \geq A/(A+B).$$

**Problem 7.** Jimmy Butler (shooting guard for the Minnesota Twolves) makes about $p$ =30% of his three-point shot attempts. In any game, the number of three-point shots he attempts is a random variable $Z$ whose distribution is approximately Poisson with mean $\lambda = 5$. What is the probability that in a random game he makes exactly 2 three-point shots?

**Problem 8.** Suppose that the mean height of UC undergraduate males is 68 inches and that the standard deviation is 3 inches. Suppose also that the mean height of MIT undergraduate males is 67.5 inches and that the standard deviation is 4 inches. Suppose that 100 UC undergraduate males and 100 MIT undergraduate males are chosen at random, with replacement.

(A) What , approximately, is the probability that the average height of the 100 UC students is more than 67.7?

(B) What, approximately, is the probability that the average height of the 100 MIT students exceeds that of the 100 UC students?