STATISTICS 251
HOMEWORK ASSIGNMENT 3
DUE FRIDAY OCTOBER 13

Problem 1. *Bridge* is a card game played by 4 players sitting around a table. In each hand, each player receives 13 cards; thus, the entire deck is partitioned among the 4. Assume that all possible arrangements of cards among the 4 players are equally likely.

(i) How many different such arrangements are there?
(ii) What is the probability that in a given hand *none* of the 4 players is void $\heartsuit$?

Problem 2. Let X be a random variable that takes values in the set $\mathbb{N} = \{1, 2, 3, \dots\}$ of positive integers. Suppose that for every pair $k, m \in \mathbb{N}$,

$$P(X - k = m \,|\, X > k) = P(X = m).$$

Show that X must have a geometric distribution.

Problem 3. Let $p(k; \lambda) = \lambda^k e^{-\lambda}/k!$ be the Poisson distribution with mean parameter $\lambda > 0$. Find the *mode* of this distribution, that is, the integer k that maximizes $p(k; \lambda)$, and give a convincing explanation (i.e., a proof).

Problem 4. The following table shows[1] the distribution of the random variable *number of children* among Chicago families:

# Children	Percent
0	10
1	20
2	40
3	20
4	10

Assume that each child in a family is equally likely to be a boy or girl, independently of all others.

(a) Suppose that a *family* is chosen at random. Given that this family has at least one boy, what is the probability that it also has at least one girl?

(b) What fraction of all families have exactly one boy (and any number of girls)?

Problem 5. Let N be the number of Aces in a randomly drawn five-card poker hand. Calculate EN in at least two different ways.

Problem 6. The *hat-check* problem: Suppose that n people give their hats to a hat-check attendant, who then returns them randomly, one to each person. (Thus, all possible re-assignments of hats to people are equally likely.) Let X be the number of people who receive their own hats back.

(A) What is the probability that $X = 0$?
(B) What is EX?
(C) What is the probability that $X = 1$?

[1] This is not really accurate.

Problem 7. Let X and Y be two *independent* random variables taking nonnegative integer values.

(a) Prove that for any nonnegative integer m

$$P\{X+Y=m\} = \sum_{k=0}^{m} P\{X=k\}P\{Y=m-k\}.$$

(b) Suppose that X $\sim$Binomial(n,p) and Y $\sim$Binomial(m,p). Suppose also that X and Y are independent. What is the distribution of $X+Y$? HINT: You should be able to puzzle this out *without* using the ugly formula of part (a).

(c) Prove the following weird formula for binomial coefficients: for $k \le \min(m,n)$

$$\sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j} = \binom{n+m}{k}.$$

(d) Let X $\sim$Binomial$(n,\frac{1}{2})$. What is $P(X \text{ is } \textit{even})$? Try to give two essentially different explanations (i.e., proofs) of your answer.

Problem 8. For any event A in a probability space, the *indicator* of A is the random variable I_A that takes the value 1 on the event A and takes the value 0 on the event A^c. (See Pitman, sec. 3.2, p. 168ff.) Show that:

(a) $I_{A^c} = 1 - I_A$.
(b) $I_{A\cap B} = I_A I_B$.
(c) For any events $A_1, A_2, \dots, A_m$,

$$I_{\cup_{i=1}^m A_i} = 1 - \prod_{i=1}^{m}(1 - I_{A_i}).$$

(d) By taking the expectation of each side of the identity in (c), give a new derivation of the *inclusion-exclusion* principle.