STATISTICS 251
HOMEWORK ASSIGNMENT 2
DUE FRIDAY OCTOBER 6

Problem 1. You are given 3 identical boxes, each with 2 drawers. Box A has a gold coin in each drawer; box B has a silver coin in each drawer; and box C has a gold coin in one drawer and a silver coin in the other. You do not know which is box A, which is box B, or which is box C, nor do you know which of the 2 drawers of box C holds the silver coin and which holds the gold. You randomly choose one of the boxes and open one of its two drawers. Suppose that it contains a gold coin. What is the conditional probability that you have chosen box A?

Problem 2. A woman has n keys, of which only one will open her door.

(a) If she tries the keys at random, discarding those that do not work, what is the probability that she will open the door on her kth try?

(b) Suppose now that she does not discard previously tried keys. What is the probability that she will open her door for the first time on her kth try?

Problem 3. Balls are randomly removed, one at a time, from an urn initially containing 20 red and 30 blue balls. What is the probability that all of the red balls are removed before all of the blue ones have been removed? HINT: Here is a similar problem. Suppose that the cards of a well-shuffled deck of 52 cards are dealt, one at a time. What is the probability that all of the $\heartsuit$, $\diamondsuit$, and $\clubsuit$ are dealt before all of the $\spadesuit$ have been dealt?

Problem 4. In baseball's annual World Series, two teams play a sequence of up to seven games, with the series ending when one of the teams has won four games. Suppose that if team A plays team B, team A has probability $.55$ of defeating team B in any particular game, and that the outcomes of successive games are independent.[1]

(a) What is the probability that team A wins the series in 5 games?
(b) What is the conditional probability that team A wins the series given that team B wins the first 2 games?

Problem 5. Let E and F be mutually exclusive events in some experiment. Suppose that independent trials of this experiment are performed repeatedly.

(a) What is the probability that event E occurs for the first time on the nth trial? What is the probability that $E \cup F$ occurs for the first time on the nth trial?

(b) What is the conditional probability that event E occurs for the first time on the nth trial *given* that event $E \cup F$ occurs for the first time on the nth trial?

(c) Show that E will occur before F with probability $\frac{\mathbb{P}(E)}{\mathbb{P}(E)+\mathbb{P}(F)}$.

[1] These assumptions are not very realistic, in part because they ignore the effect of home field advantage, and also because a team's best starting pitcher cannot be used in every game.

Problem 6. Let $a_1, a_2, \ldots, a_n$ be a random ordering of the integers $1, \ldots, n$, and let $b_1, b_2, \ldots, b_n$ be a random sequence where each b_i is obtained by random sampling from the integers $1, \ldots, n$ (with replacement).

(a) What is the probability r_n that there exists no $1 \leq i \leq n$ such that $b_i = i$?
(b) What is the probability s_n that there exists no $1 \leq i \leq n$ such that $a_i = i$? HINT: Inclusion-Exclusion.
(c) Compute the limits of the sequences r_n and s_n in Parts (a) and (b) when $n \to \infty$.

Problem 7. Toss a fair coin n times.

(a) Let $n = 8$. Calculate the probability that two consecutive Heads do not occur in the sequence of coin tosses.

(b) Let f_n be the number of different outcomes (i.e., sequences of H and T of length n) in which there are *no consecutive Hs*, that is, every H is follows by at least one T. Show that f_n satisfies the recursive relationship

$$f_{n+2} = f_{n+1} + f_n.$$

Problem 8. How might you *simulate* a roll of a fair die if you only had available a fair coin? You can toss the coin as many times as you like, but of course it would be preferable to toss it as few times as possible.