Suggested reading in the 4th edition of Strang's *Linear Algebra and Its Applications*:

Section 3.1, 3.2 and 3.3.

Suggested problems: From the 8 listed below, please select 6. Unstarred problems are worth 8 points; starred problems are worth 12. *You may not refer to notes from discussions with peers or professors while preparing the solutions you plan to submit.*

1. *Strang* Section 3.1: 8 and 42.
2. *Strang* Section 3.2: 8 and 16.
3. *Strang* Section 3.3: 6 and 8.
4. *Strang* Section 3.3: 18 and 32.

5^*. Suppose that $H_1, H_2, \ldots, H_k$ are k hyperplanes in $\mathbb{R}^n$, where $n \geq 2$. (Recall that a *hyperplane* is a vector subspace of $\mathbb{R}^n$ with dimension $n-1$.) Let $W = \cap_{i=1}^k H_k$ be the intersection of the hyperplanes H_i.

(a) Show (briefly) that W is a vector subspace of $\mathbb{R}^n$.

(b) Show that there are unit vectors $v_1, v_2, \ldots, v_k$ such that v_i is orthogonal to H_i for each $i = 1, 2, \ldots, k$.

(c) Show that $W^\perp$ =span$\{v_1, v_2, \ldots, v_k\}$.

6^*. The Schwarz inequality for vectors in an inner product space[1] asserts that for any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\|^2 \|\mathbf{v}\|^2,$$

in other words, the square root of the dot product of two vectors is no larger than the product of their lengths. (NOTE: The Schwartz inequality is essentially equivalent to the fact that the *correlation* between two random variables is always between -1 and +1. We'll talk about this in class soon.)

(a) Prove the Schwarz inequality. Under what conditions will equality hold? HINT: Start by writing out the inner product of $\mathbf{u}+\mathbf{v}$ with itself, and use the bilinearity and symmetry of the inner product. This should get you halfway there.

(b) Use the result in (a) to establish the following fact. For real numbers $x_1, x_2, \ldots, x_n$,

$$\left(\sum_{i=1}^n x_i\right)^2 \leq n \left(\sum_{i=1}^n x_i^2\right).$$

Under what conditions does equality hold?

7^*. Consider the following data points: $(36, 88)$, $(67.25, 224.7)$, $(93, 365.3)$, $(141.75, 687)$, and $(483.8, 4332.1)$. (This was the data Kepler had when he was trying to find a relationship between the distance, in millions of miles, of a planet from the sun and the time, in days,

[1] An inner product space is a vector space on which there is an inner product. There is no harm in assuming that it is just $\mathbb{R}^n$ with the usual dot product.

it takes for that planet to go once around the sun.) Rather than fit a line $y = b + mx$ to these data points (x_i, y_i), fit a line to the points $(\log x_i, \log y_i)$. Use your fitted line to guess a simple relationship between x and y. (In many applications to science and social science, data is transformed before regression is employed. The most commonly used transformations are power tranformations and logarithmic transformations.)

8^*. Recall that a real $n \times n$ matrix is an *orthogonal* matrix if its columns form an orthonormal basis of $\mathbb{R}^n$.

(a) Show that Q is an orthogonal matrix if and only if $Q^T Q = I$ where I is the $n \times n$ identity matrix.

(b) Show that Q is an orthogonal matrix if and only if for every pair of vectors $v, w \in \mathbb{R}^n$,

$$\langle Qv, Qw \rangle = \langle v, w \rangle$$

Thus, orthogonal matrices preserve lengths.

A *complex* $n \times n$ matrix U is called a *unitary* matrix if its columns form an orthonormal basis of $\mathbb{C}^n$ (relative to the inner product defined in class).

(c) Show that U is unitary if and only if $U^*U = I$, where U^* is the *conjugate transpose* of U, that is, if $U = (a_{i,j})$ then $U^* = (\bar{a_{j,i}})$.

(d) Show that U is unitary if and only if for every pair of vectors $u, v \in \mathbb{C}^n$,

$$\langle Uv, Uw \rangle = \langle v, w \rangle$$

(e) Let $U = (c_{j,k})$ be the $n \times n$ complex matrix whose j, k th entry is

$$c_{j,k} = \exp\{2\pi i jk/n\}$$

Show that U is unitary.