Suggested reading in the 4th edition of Strang's *Linear Algebra and Its Applications*:

Sections 1.1, 1.2, 1.3 and 1.4.

Suggested problems: From the 8 listed below, please select 6 that you find instructive. Unstarred problems are worth 8 points each; starred problems are worth 12 points each. If you submit solutions to more than 6 problems, we will select the best. The total number of points for this assignment is 64. For you to abide by the University Policy on Academic Honesty and Plagiarism, I expect you to avoid accidental plagiarism by following the collaboration guidelines detailed in the course syllabus. In particular, *you may not refer to notes from discussions with peers or professors while preparing the solutions you plan to submit.*

1. Demand d and supply s of three goods, A B C, as functions of prices, p_A, p_B and p_C, in a purely competetive market are:

$$\begin{aligned} d_A &= 100 - 2p_A - 2p_B + p_C & s_A &= 3p_A - 65 \\ d_B &= 135 - 4p_A - 3p_B + 2p_C & s_B &= 5p_B - 95 \\ d_C &= 140 + p_A + p_B - 2p_C & s_C &= 6p_C - 10 \end{aligned}$$

Hence, at equilibrium, quantity demanded equals quantity supplied for each of the three goods.

$$\begin{aligned} 100 - 2p_A - 2p_B + p_C &= 3p_A - 65 \\ 135 - 4p_A - 3p_B + 2p_C &= 5p_B - 95 \\ 140 + p_A + p_B - 2p_C &= 6p_C - 10 \end{aligned}$$

Determine the equilibrium prices and quantities of the three goods in this competitive market.

2. This problem is based on the true story of a Mennonite farmer who, in striving for energy self-sufficiency, hit upon the idea of distilling his own alcohol to run his tractor. To grow one bushel of corn he requires K liters of alcohol. To produce one liter of alcohol he requires L bushels of corn, plus M liters of alcohol to fire the still. How many bushels of corn and how many liters of alcohol must he produce in order to run his tractor and his still, and also have N bushels of corn left over to feed his animals? (Your answers should be given in terms of the constants K, L, M, and N. State any assumptions you need to make about these variables so that your solution is valid.)

3. *Strang* Section 1.4: Exercise 14 and 16.

4. *Strang* Section 1.4: Exercises 24 and 40.

5*. (a) Find the set of all solutions to the system of 3 equations in 4 unknowns that is summarized by the matrix equation below:

$$\begin{bmatrix} 1 & -7 & 0 & 6 \\ 0 & 0 & 1 & -2 \\ -1 & 7 & -4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 7 \end{bmatrix}$$

The set of points (x_1, x_2, x_3, x_4) that satisfy all three equations is the intersection of three hyperplanes in $\mathbf{R}^4$. Is the intersection 1-dimensional (a line)? If so, give a point on the line and a vector parallel to the line. Is the intersection 2-dimensional? If so, give a point in the 2-dimensional intersection and 2 vectors (neither of which is a multiple of the other) that are parallel to the 2-dimensional intersection.

(b) Find the *interpolating polynomial* $p(t) = a_0 + a_1 t + a_2 t^2$ for the data (1,12), (2,15), (3,16). That is, find a_0, a_1, and a_2 such that

$$\begin{aligned} a_0 + a_1(1) + a_2(1)^2 &= 12 \\ a_0 + a_1(2) + a_2(2)^2 &= 15 \\ a_0 + a_1(3) + a_2(3)^2 &= 16 \end{aligned}$$

Is the graph of p the only parabola that passes through these 3 points? More generally, given any 3 points in 2-space, is there always exactly one polynomial of degree 2 whose graph passes through them? Be careful. For example, do the first coordinates of the points have to be 1, 2 and 3 for there to be exactly one "interpolating" polynomial of degree 2? Could an interpolating polynomial have degree 1 for a different collection of 3 points? Can you formulate a precise conjecture about when there is a unique interpolating polynomial to 3 points? To more than 3 points?

6*. (a) Use vectors to prove that, for any triangle, the three line segments that each join a vertex to the midpoint of the opposite side meet at a point. Prove that this point, called the *centroid* of the triangle, is 2/3 the distance from each vertex along the line segment to the opposite midpoint.

(b) Prove that for any tetrahedron the four line segments that each join a vertex to the centroid of the opposite face meet in a point. Into what ratio does that point divide each line segment?

(c) In a 2-dimensional Euclidean space, any three points that do not lie along the same line are the vertices of a triangle. This triangle is called the 2-simplex whose vertices are the three points. In a 3-dimensional Euclidean space, any four points that do not lie in the same plane are the vertices of a tetrahedron. This tetrahedron is called the 3-simplex whose vertices are the four points. Generalize the results in (a) and (b) to an n-simplex in n-dimensional Euclidean space, where $n \geq 4$.

7*. A vector in $\mathbf{C}^2$ is a list $\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ of two complex numbers z_1 and z_2. Define the inner product of two vectors $\mathbf{z}$ and $\mathbf{w}$ in $\mathbf{C}^2$ by $\langle \mathbf{z}, \mathbf{w} \rangle = z_1 \overline{w_1} + z_2 \overline{w_2}$. Prove the following properties of this inner product: $\langle \mathbf{z}, \mathbf{w} \rangle = \overline{\langle \mathbf{w}, \mathbf{z} \rangle}$; $\langle \mathbf{u} + \mathbf{w}, \mathbf{z} \rangle = \langle \mathbf{u}, \mathbf{z} \rangle + \langle \mathbf{w}, \mathbf{z} \rangle$; $\langle c\mathbf{z}, \mathbf{w} \rangle = c \langle \mathbf{z}, \mathbf{w} \rangle$ for any complex number c; $\langle \mathbf{z}, \mathbf{z} \rangle \geq 0$ for all vectors $\mathbf{z}$ and equals 0 if and only if $\mathbf{z} = \mathbf{0}$. Define an inner product on $\mathbf{C}^n$ that generalizes this inner product on $\mathbf{C}^2$.

8*. Define two vectors $\mathbf{z}$ and $\mathbf{w}$ in $\mathbf{C}^n$ to be orthogonal if (and, of course, only if) $\langle \mathbf{z}, \mathbf{w} \rangle = 0$. Define the length of a vector $\mathbf{z}$ in $\mathbf{C}^n$ by $\|\mathbf{z}\| = \sqrt{\langle \mathbf{z}, \mathbf{z} \rangle}$. Prove that if $\mathbf{z}$ and $\mathbf{w}$ are orthogonal, then $\|\mathbf{z}\|^2 + \|\mathbf{w}\|^2 = \|\mathbf{z} + \mathbf{w}\|^2$. Is it true that if $\|\mathbf{z}\|^2 + \|\mathbf{w}\|^2 = \|\mathbf{z} + \mathbf{w}\|^2$ then $\mathbf{z}$ and $\mathbf{w}$ are orthogonal? If so, prove it; if not, provide a counterexample.