

Research Statement on Mathematics

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My research on mathematics during PhD period is in the area of algebraic geometry, geometric invariant theory(GIT) and differential geometry. I focused on desingularizations of GIT quotients using Kirwan's general theory of stratification, concentrating on moduli spaces of cubic surfaces and Del Pezzo surfaces.

Thesis Work on Algebraic Geometry

The homogeneous cubic polynomials in four variables x, y, z and w form a 20 dimensional vector space over \mathbb{C} . Let $\mathcal{S} \subset \mathbb{P}^{19}$ be a surface defined by the equation $f(x, y, z, w) = a_0x^3 + a_1x^2y + \dots + a_{19}w^3 = 0$, which naturally determines a point $[a_0, \dots, a_{19}] \in \mathbb{P}^{19}$. X admits a natural $G = SL(4; \mathbb{C})$ action. Let X^{sm} denote the smooth cubic surfaces, X^{ss} the semistable ones and X^s the stable ones. Two cubic surfaces are isomorphic if one can be transformed to the other by some element of $SL(4; \mathbb{C})$. Let $R = \mathbb{C}[a_0, \dots, a_{19}]^G$ be the invariant graded subring, which is finitely generated, and let $\overline{\mathcal{M}} = Proj R$ a projective variety. GIT yields a surjective morphism $X^{ss} \rightarrow \overline{\mathcal{M}}$. Let X^{sm} denote the isomorphism classes of smooth cubic surfaces and $\overline{\mathcal{M}}^s$ the isomorphism classes of stable cubic surfaces, i.e, the cubic surfaces with at worst nodal singularities locally isomorphic to $x^2 + y^2 = z^2$. The bad news is that $\overline{\mathcal{M}}$ does not have a moduli interpretation, i.e, as the isomorphism classes of cubic surfaces with prescribed singularities. In the thesis, I give a geometric compactification of \mathcal{M}^s .

PART I: Naruki's cross ratio variety

By a marked smooth cubic surface S , we mean a smooth cubic surface S together with a marking of the 27 lines l_1, \dots, l_{27} on it. Let \mathcal{M}_{mark}^{sm} denote the moduli space of marked smooth cubic surfaces and \mathcal{M}_{mark} denote the normalization of \mathcal{M} in the function field of the 27 lines. The projection from each line l on S gives a conic bundle $\pi : S \rightarrow \mathbb{P}^1$ which has 5 degenerate fibres consisting of two intersecting lines, say over $P_1, P_2, P_3, P_4, P_5 \in \mathbb{P}^1$. Any fixed ordered 4-tuple (P_i, P_j, P_k, P_m) gives a cross ratio function $r(P_i, P_j, P_k, P_m) = \frac{(P_i - P_k)(P_j - P_m)}{(P_j - P_k)(P_i - P_m)}$. There are 270 distinct cross ratios and they define a morphism $\lambda_s : \mathcal{M}_{mark}^{sm} \rightarrow (\mathbb{P}^1)^{270}$, which is actually an embedding. After analyzing all the degenerate fibres of conic bundles associated to nodal cubic surfaces, I prove the following fact, stated without proof by Naruki.

Theorem 1. The cross ratio map λ_s extends to a morphism $\lambda : \mathcal{M}_{mark}^s \rightarrow (\mathbb{P}^1)^{270}$ uniquely.

Definition. Naruki's marked cross ratio variety \mathcal{C}_{mark} is defined as the closure of the image of λ and correspondingly, cross ratio variety

$\mathcal{C} = \mathcal{C}_{mark}/W(E_6)$. Here the Weyl group $W(E_6)$ is the automorphism group of the configuration of the 27 lines on a smooth cubic surface S .

Naruki showed that the cross ratio variety \mathcal{C}_{mark} is nonsingular and the boundary $\mathcal{C}_{mark} - \mathcal{M}_{mark}^s$ is a divisor with normal crossings. Therefore, \mathcal{C}_{mark} a priori is not a moduli space. In this thesis, we show that the points of $\mathcal{C}_{mark} - \mathcal{M}_{mark}^s$ correspond to certain degenerations of cubic surfaces, called triploid surfaces. These triploid surfaces explicitly constructed starting with 3 sets of 4 distinguished points on $\mathbb{P}^1 \times \mathbb{P}^1$ by blowing up, blowing down and glueing operations. In particular, the cross ratio functions can be defined for these surfaces. The projective geometry of triploid surfaces is easy to describe. The image of a smooth cubic surface S under the degree 2 Veronese embedding $v_2 : S \subset \mathbb{P}^3 \rightarrow \mathbb{P}^9$ has degree 12. Triploid surfaces are explicit degenerations of $v_2(S) \subset \mathbb{P}^9$.

Theorem 2. (Main Theorem I) The cross ratio variety \mathcal{C} has a moduli interpretation as the isomorphism classes of stable cubic surfaces and triploid surfaces.

Theorem 3. (i) A triploid surface has 27 conics consisting of two intersecting lines; they are the flat limits of the 27 lines on a smooth cubic surface. (ii) There are 18 cross ratios associated to a triploid surface.

Part II: Kirwan's stratification and blowing up

Suppose $X \subset \mathbb{P}^n$ is a nonsingular projective variety with a reductive group G acting linearly. Kirwan gave a G -invariant stratification $X = \mathcal{S}_0 \cup \{\mathcal{S}_\beta | \beta \neq 0\}$, with the unique open stratum \mathcal{S}_0 coinciding with X^{ss} and other strata $\{\mathcal{S}_\beta | \beta \neq 0\}$ indexed by the 1-parameter subgroup β of G which is most responsible for \mathcal{S}_β being unstable. This stratification is used to compute the cohomology of the quotient $X//G$. When $X^{ss} = X^s$, the equivariant Poincare polynomial $P_t^G(X^{ss}) = P_t(X^{ss}//G)$ can be obtained from $P_t^G(X)$ by subtracting off the contribution $P_t^G(\mathcal{S}_\beta)$ of each unstable stratum \mathcal{S}_β .

However, in many interesting cases X^{ss} does not coincide with X^s . Kirwan developed a systematic way of blowing up X along a sequence of nonsingular subvarieties to obtain a variety \tilde{X} with a lifted linear action of G such that every semistable point of \tilde{X} is stable.

In the case of cubic surfaces, we have

Theorem 4. To resolve the strictly semistable locus, it suffices to blow up once along the $SL(4; \mathbb{C})$ -orbit of toric cubic surface $\{w^3 - xyz = 0\}$. The modified formula for the rational cohomology of $\tilde{X}//G$ gives

$$P_t(\widetilde{\mathbb{P}^{19}}//G) = 1 + 2t^2 + 2t^4 + 2t^6 + t^8.$$

However, Naruki showed that the boundary $\mathcal{C}_{mark} - \mathcal{M}_{mark}^s$ has 36 nodal divisors which parametrize nodal cubic surfaces and 40 disjoint cusp divisors, each of which after blowing down looks like the vertex of the cone

over the Veronese imbedding $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ into \mathbb{P}^7 . This together with an analysis of the normal slice of toric cubic surface motivates me to believe that the cross ratio variety \mathcal{C} is obtained by a weighted Kirwan blow-up of $\overline{\mathcal{M}}$ at the toric point.

Theorem 5. (Main Theorem II) There is a birational morphism $k : \mathcal{C} \rightarrow \overline{\mathcal{M}}$ which is induced by a weighted Kirwan blow up at the point corresponding to the orbit of toric cubic surfaces.

Actually among all the strictly semi-stable cubic surfaces, the only closed G -orbit is the orbit of the toric cubic surface. The orbit closure of any other strictly semi-stable cubic surface with a cuspidal singularity contains this toric surface.

Even more, using a weighted version of Kirwan's blowing-up formula and the rational cohomology (which actually is the Chow ring) of \mathcal{C} , one can write out a basis of the Chow ring and compute explicitly the Chow group $\mathcal{A}^i(\mathcal{C})$ and Chern classes of \mathcal{C} , and the classes and intersection numbers of the 36 nodal divisors, 40 cusp divisors, and the 45 divisors which parametrize the cubic surfaces with an Eckardt point. This approach recovers the results obtained by Van Geemen and Elisabetta using Naruki's explicit construction of the cross ratio variety.

Planned Research on Geometric Invariant Theory

I. Weighted Pointed Stable Curves

Let n be a positive integer and let $\mathcal{A} = (a_1, \dots, a_n) \in \mathbb{Q}^n$ such that $0 < a_j \leq 1$ for $j = 1, \dots, n$ and $a_1 + \dots + a_n > 2$. A family of rational curves with marked points $\pi : (C, s_1, \dots, s_n) \rightarrow B$ is \mathcal{A} -stable if the sections s_1, \dots, s_n of π lie in the smooth locus of π , for any subset $\{s_{i_1}, \dots, s_{i_r}\}$ with $s_{i_1}(b) = \dots = s_{i_r}(b)$ for some $b \in B$ we have $a_{i_1} + \dots, a_{i_r} \leq 1$, and $K_\pi + a_1 s_1 + \dots + a_n s_n$ is π -relatively ample.

Theorem (Hassett) There exists a connected scheme $\overline{\mathcal{M}}_{0,\mathcal{A}}$, smooth and projective over \mathbb{Z} , representing the moduli problem of pointed \mathcal{A} -stable rational curves.

Hassett interpreted the intermediate steps in various constructions of the moduli spaces $\overline{\mathcal{M}}_{0,n}$ as certain moduli spaces $\overline{\mathcal{M}}_{0,\mathcal{A}}$ and studied how GIT quotients of $(\mathbb{P}^1)^n$ may be interpreted as small-parameter limits of the $\mathcal{M}_{0,\mathcal{A}}$ as $\sum_{j=1}^n a_j \rightarrow 2$. In particular, there is a birational morphism $h : \mathcal{M}_{0,\mathcal{A}} \rightarrow (\mathbb{P}^1)^n // SL_2$, where $(\mathbb{P}^1)^n // SL_2$ is the GIT quotient with respect to the linearization $\mathcal{O}_{(\mathbb{P}^1)^n}(a_1, \dots, a_n)$. We expect to prove the following:

Claim: The birational morphism $h : \mathcal{M}_{0,\mathcal{A}} \rightarrow (\mathbb{P}^1)^n // SL_2$ can be interpreted as the Kirwan weighted blow-up along closed orbits in the strictly semistable locus.

II. Quartic Del Pezzo Surfaces

A del Pezzo orbifold S is an irreducible compact complex surface, with only isolated quotient singularities and positive first Chern class. The anticanonical image of a smooth $S \subset \mathbb{P}^4$ can be realized as the intersection of two hyperquadrics $q = 0$ and $r = 0$, where q and r are two quadratic forms. The point of the Grassmannian $G(2, S^2(\mathbb{C}^5))$ corresponding to the complex 2-plane spanned by q and r is called the Hilbert point of S . If S is nonsingular, its Hilbert point is stable with respect to the action of $SL(5; \mathbb{C})$. The GIT quotient of $G(2, S^2(\mathbb{C}^5))$ under the action of $SL(5; \mathbb{C})$ gives a compactification of the set Σ of isomorphism classes of smooth quartic del Pezzo surfaces. However, Mabuchi and Mukai prove that quartic del Pezzo orbifolds with only nodes as singularities are strictly semistable and admit an Einstein-Kähler metric. In fact, the Chow points of these singular quartic del Pezzo surfaces are also not stable. Is there a natural GIT quotient that yields a geometric compactification of the moduli space of quartic del Pezzo surfaces?

Apply the Kirwan stratification and blow up along the closed orbits in the strictly semistable locus, i.e., both Hilbert points and Chow points, to obtain a compact moduli space of quartic Del Pezzo surfaces and a geometric interpretation of this compactification.