Bernoulli Distribution

Example: Toss of coin

Define $X = 1$ if head comes up and
$X = 0$ if tail comes up.
Both realizations are equally likely: $\mathbb{P}(X = 1) = \mathbb{P}(X = 0) = \frac{1}{2}$

Examples:

Often: Two outcomes which are *not* equally likely:

- Success of medical treatment
- Interviewed person is female
- Student passes exam
- Transmittance of a disease

Bernoulli distribution (with parameter θ)

- X takes two values, 0 and 1, with probabilities p and $1 - p$
- Frequency function of X

$$p(x) = \begin{cases} \theta^x (1-\theta)^{1-x} & \text{for } x \in \{0,1\} \\ 0 & \text{otherwise} \end{cases}$$

- Often:

$$X = \begin{cases} 1 & \text{if event } A \text{ has occured} \\ 0 & \text{otherwise} \end{cases}$$

Example: A = blood pressure above 140/90 mm HG.

Bernoulli Distribution

Let $X_1, \dots, X_n$ be independent Bernoulli random variables with same parameter θ.

Frequency function of $X_1, \dots, X_n$

$$p(x_1, \dots, x_n) = p(x_1) \cdots p(x_n) = \theta^{x_1 + \dots + x_n} (1-\theta)^{n - x_1 - \dots - x_n}$$

for $x_i \in \{0, 1\}$ and $i = 1, \dots, n$

Example: Paired-Sample Sign Test

- Study success of new elaborate safety program
- Record average weekly losses in hours of labor due to accidents before and after installation of the program in 10 industrial plants

Plant	1	2	3	4	5	6	7	8	9	10
Before	45	73	46	124	33	57	83	34	26	17
After	36	60	44	119	35	51	77	29	24	11

Define for the ith plant

$$X_i = \begin{cases} 1 & \text{if first value is greater than the second} \\ 0 & \text{otherwise} \end{cases}$$

Result: 1 1 1 1 0 1 1 1 1 1

The X_i's are *independently Bernoulli distributed* with unknown parameter θ.

Binomial Distribution

Let $X_1, \ldots, X_n$ be independent Bernoulli random variables

- Often only interested in number of successes

$$Y = X_1 + \ldots + X_n$$

Example: Paired Sample Sign Test (contd)

Define for the ith plant

$$X_i = \begin{cases} 1 & \text{if first value is greater than the second} \\ 0 & \text{otherwise} \end{cases}$$

$$Y = \sum_{i=1}^{n} X_i$$

Y is the number of plants for which the number of lost hours has decreased after the installation of the safety program

We know:

- X_i is Bernoulli distributed with parameter θ
- X_i's are independent

What is the distribution of Y*?*

- Probability of realization $x_1, \ldots, x_n$ with y successes:

$$p(x_1, \ldots, x_n) = \theta^y (1 - \theta)^{n-y}$$

- Number of different realizations with y successes: $\binom{n}{y}$

Binomial Distribution

Binomial distribution (with parameters n and θ)

Let $X_1, \ldots, X_n$ be independent and Bernoulli distributed with parameter θ and

$$Y = \sum_{i=1}^{n} X_i.$$

Y has *frequency function*

$$p(y) = \binom{n}{y} \theta^y (1-\theta)^{n-y} \qquad \text{for } y \in \{0, \ldots, n\}$$

Y is *binomially distributed* with parameters n and θ. We write

$$Y \sim \mathrm{Bin}(n, \theta).$$

Note that

- the number of trials is fixed,
- the probability of success is the same for each trial, and
- the trials are independent.

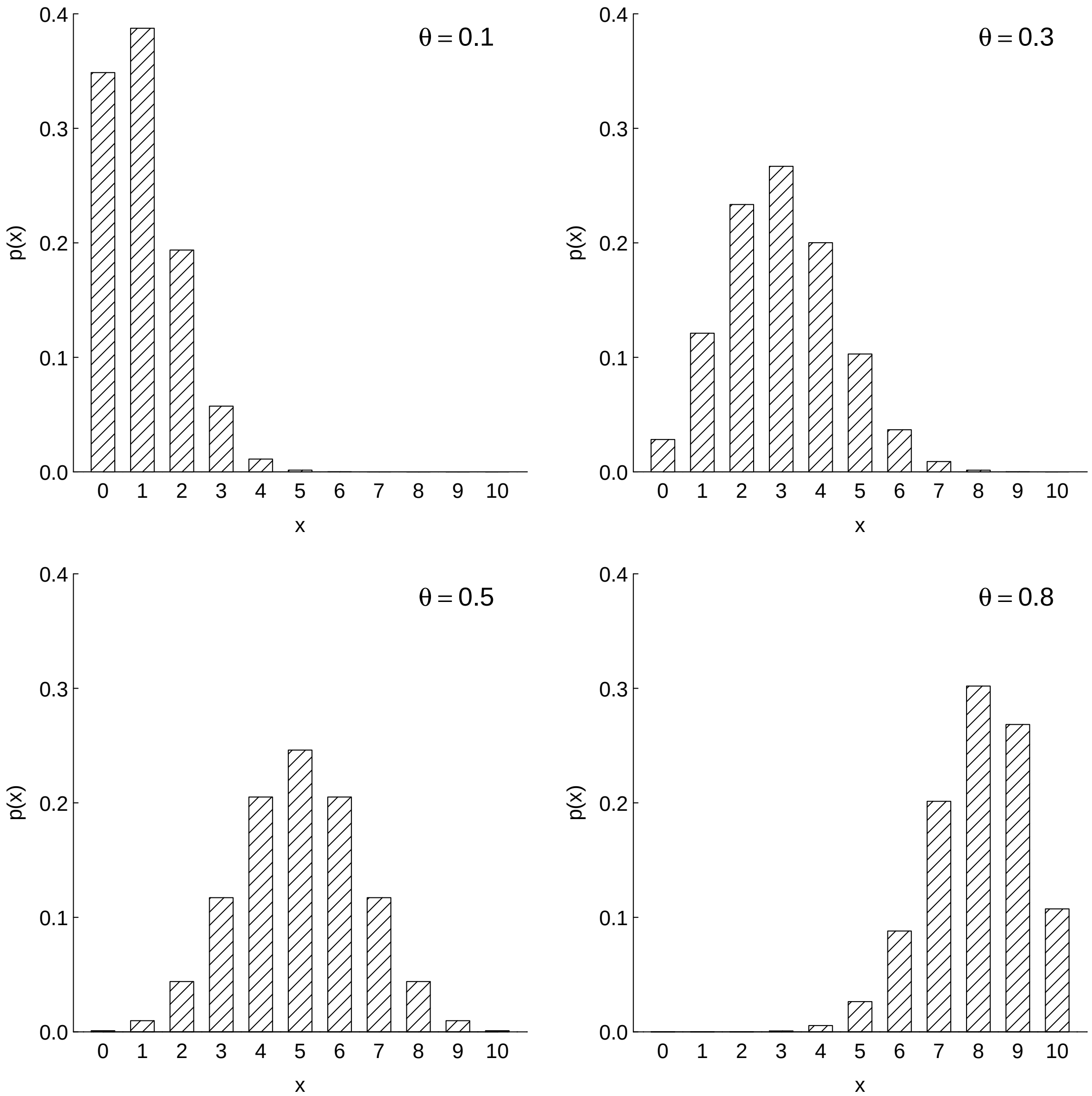
Example: Paired Sample Sign Test (contd)

Let Y be the number of plants for which the number of lost hours has decreased after the installation of the safety program. Then

$$Y \sim \mathrm{Bin}(n, \theta)$$

Binomial Distribution

Binomial distribution for $n = 10$



Geometric Distribution

Consider a sequence of independent Bernoulli trials.

- On each trial, a success occurs with probability θ.
- Let X be the number of trials up to the first success.

What is the distribution of X?

- Probability of no success in $x-1$ trials: $(1-\theta)^{x-1}$
- Probability of one success in the xth trial: θ

The frequency function of X is

$$p(x) = \theta(1-\theta)^{x-1}, \qquad x = 1, 2, 3, \ldots$$

X is *geometrically distributed* with parameter θ.

Example:

Suppose a batter has probability $\frac{1}{3}$ to hit the ball. What is the chance that he misses the ball less than 3 times?

The number X of balls up to the first success is geometrically distributed with parameter $\frac{1}{3}$. Thus

$$\mathbb{P}(X \leq 3) = \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} + \frac{1}{3}\Big(\frac{2}{3}\Big)^2 = 0.7037.$$

Hypergemetric Distribution

Example: Quality Control

Quality control - sample and examine fraction of produced units

- N produced units
- M defective units
- n sampled units

What is the probability that the sample contains x defective units?

The frequency function of X is

$$p(x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}, \qquad x = 0, 1, \ldots, n.$$

X is a *hypergeometric* random variable with parameters N, M, and n.

Example:

Suppose that of 100 applicants for a job 50 were women and 50 were men, all equally qualified. If we select 10 applicants at random what is the probability that x of them are female?

The number of chosen female applicants is hypergeometrically distributed with parameters 100, 50, and 10. The frequency function is

$$p(x) = \frac{\binom{50}{x}\binom{50}{10-x}}{\binom{100}{10}} \qquad \text{for } x \in \{0, \ldots, n\}$$

for $x = 0, 1, \ldots, 10$.

Poisson Distribution

Often we are interested in the number of events which occur in a specific period of time or in a specific area of volume:

- Number of alpha particles emitted from a radioactive source during a given period of time
- Number of telephone calls coming into an exchange during one unit of time
- Number of diseased trees per acre of a certain woodland
- Number of death claims received per day by an insurance company

Characteristics

Let X be the number of times a certain event occurs during a given unit of time (or in a given area, etc).

- The probability that the event occurs in a given unit of time is the same for all the units.
- The number of events that occur in one unit of time is independent of the number of events in other units.
- The mean (or expected) rate is λ.

Then X is a *Poisson random variable* with parameter λ and frequency function

$$p(x) = \frac{\lambda^x}{x!}\, e^{-\lambda}, \qquad x = 0, 1, 2, \ldots$$

Poisson Approximation

The *Poisson distribution* is often used as an approximation for binomial probabilities when n is large and θ is small:

$$p(x) = \binom{n}{x} \theta^x (1-\theta)^{n-x} \approx \frac{\lambda^x}{x!} e^{-\lambda}$$

with $\lambda = n\,\theta$.

Example: Fatalities in Prussian cavalry

Classical example from von Bortkiewicz (1898).

- Number of fatalities resulting from being kicked by a horse
- 200 observations (10 corps over a period of 20 years)

Statistical model:

- Each soldier is kicked to death by a horse with probability θ.
- Let Y be the number of such fatalities in one corps. Then

  $$Y \sim \text{Bin}(n, \theta)$$

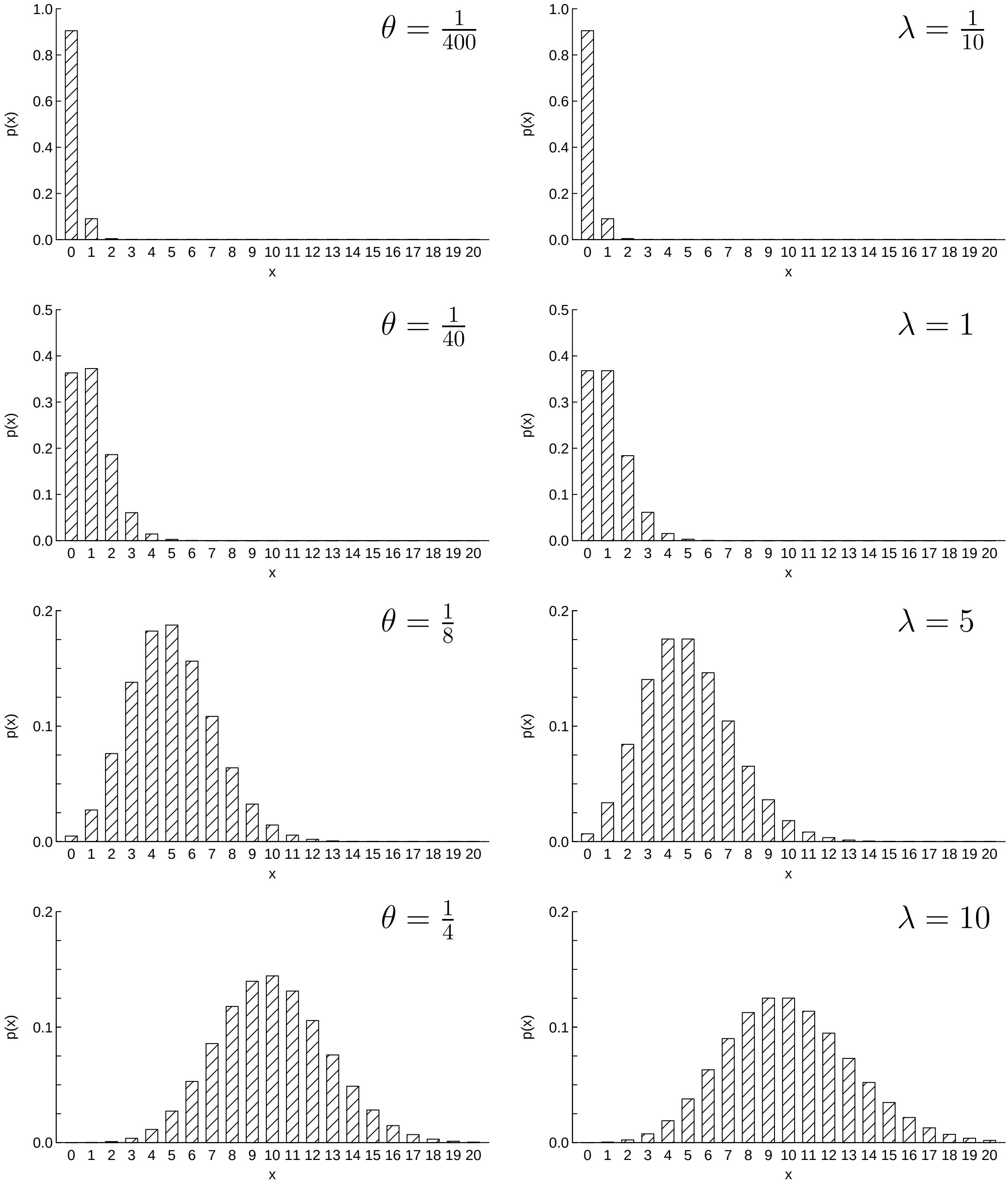
  where n is the number of soldiers in one corps.

Observation: The data are well approximated by a Poisson distribution with $\lambda = 0.61$

Deaths per Year	Observed	Rel. Frequency	Poisson Prob.
0	109	0.545	0.543
1	65	0.325	0.331
2	22	0.110	0.101
3	3	0.015	0.021
4	1	0.005	0.003

Poisson Approximation

Poisson approximation of Bin$(40, \boldsymbol{\theta})$



Continuous Distributions

Uniform distribution $U(0, \theta)$

Range $(0, 1)$

$f(x) = \frac{1}{\theta} 1_{(0,\theta)}(x)$

$\mathbb{E}(X) = \frac{\theta}{2}$

$\text{var}(X) = \frac{\theta^2}{12}$

Exponential distribution $\text{Exp}(\lambda)$

Range $[0, \infty)$

$f(x) = \lambda \exp(-\lambda x) 1_{[0,\infty)}(x)$

$\mathbb{E}(X) = \frac{1}{\lambda}$

$\text{var}(X) = \frac{1}{\lambda^2}$

Normal distribution $\mathcal{N}(\mu, \sigma^2)$

Range $\mathbb{R}$

$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$

$\mathbb{E}(X) = \mu$

$\text{var}(X) = \sigma^2$