Fourier analysis on the symmetric group

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Fourier transform on $\mathbb{S}_n$

Forward transform:

$$\widehat{f}(\lambda) = \sum_{\sigma \in \mathbb{S}_n} f(\sigma)\, \rho_\lambda(\sigma) \qquad\qquad \lambda \vdash n$$

Inverse transform:

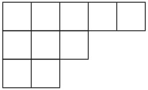
$$f(\sigma) = \frac{1}{n!} \sum_{\lambda \vdash n} d_\lambda \operatorname{tr}\big[\widehat{f}(\lambda)\, \rho_\lambda(\sigma^{-1})\big].$$

Integer partitions

$\lambda \vdash n$ means that $\lambda = (\lambda_1, \ldots, \lambda_k)$ is an integer partition of n, i.e.,

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \qquad \text{and} \qquad \sum_{i=1}^{k} \lambda_i = n.$$

Graphically represented by so-called **Young diagrams** (Ferrers diagrams), such as



for $\lambda = (5, 3, 2) \vdash 10$.

Fact: The irreps of $\mathbb{S}_n$ are indexed by $\{\lambda \vdash n\}$.

Young tableaux

Filling in the rows/columns of a Young diagram with $1, \ldots, n$ gives a so-called **Young tableau**.

A **standard tableau** is a Young tableau in which the numbers strictly increase from left to right in each row, and top to bottom in each column. Example:

1	3	5	6
2	4		
7			

The set of standard tableaux of shape λ we'll denote $\mathcal{T}_\lambda$.

Fact: The rows/columns of $\rho_\lambda(\sigma)$ are in bijection with $\mathcal{T}_\lambda$.

Hook rule

Given any box i in a Young diagram, a hook is that box, plus the boxes to the right, plus the boxes below. The length $\ell(i)$ of the hook is the total number of boxes involved.

Theorem.

$$|\,\mathcal{T}_\lambda\,| = d_\lambda = \frac{n!}{\prod_i \ell(i)}.$$

Corollary.

$$\sum_{\lambda \vdash n} d_\lambda^2 = n!$$

λ	d_λ
(n)	1
$(n-1,1)$	$n-1$
$(n-2,2)$	$\frac{n(n-3)}{2}$
$(n-2,1,1)$	$\frac{(n-1)(n-2)}{2}$
$(n-3,3)$	$\frac{n(n-1)(n-5)}{6}$
$(n-3,2,1)$	$\frac{n(n-2)(n-4)}{3}$
$(n-3,1,1,1)$	$\frac{(n-1)(n-2)(n-3)}{6}$
$(n-4,4)$	$\frac{n(n-1)(n-2)(n-7)}{24}$
$(n-4,3,1)$	$\frac{n(n-1)(n-3)(n-6)}{8}$
$(n-4,2,2)$	$\frac{n(n-1)(n-4)(n-5)}{12}$
$(n-4,2,1,1)$	$\frac{n(n-2)(n-3)(n-5)}{8}$

Young's Orthogonal Representation

Let τ_k be the adjacent transposition $(k, k+1)$. The action of τ_k on a tableau t is to swap the numbers k and $k+1$.

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 & \multicolumn{2}{c}{} \\ \cline{1-2} 7 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}$$

Define $c_t(i)$ as the column index minus the row index of the cell in t where i is found. Define the signed distance $d_t(i,j) = c_t(j) - c_t(i)$.

In YOR the matrix $\rho_\lambda(\tau_k)$ is defined as follows. Take any $t \in \mathcal{T}_\lambda$. In row t:

- The diagonal element is $[\rho_\lambda(\tau_k)]_{t,t} = 1/d_t(k, k+1)$
- If $\tau_k(t)$ is a standard tableau, then we also have the off-diagonal element

$$[\rho_\lambda(\tau_k)]_{t,\tau_k(t)} = \sqrt{1 - 1/d_t(k, k+1)^2}.$$

- All other elements are zero.

Young's Orthogonal Representation

- Adapted to the subgroup chain $\mathbb{S}_n > \mathbb{S}_{n-1} > \mathbb{S}_{n-2} > \ldots > \mathbb{S}_1$.
- The matrices of adjacent transpositions are very sparse (≤ 2 nonzeros in each row).

Any **contiguous cycle** $[\![i, j]\!] = (i, i+1, \ldots, j)$ can be written as a product of $j - i$ adjacent transpositions:

$$[\![i, j]\!] = \tau_i \tau_{i+1} \ldots \tau_{j-1}.$$

Any $\sigma \in \mathbb{S}_n$ can be written as a product of at most $n - 1$ contiguous cycles (insertion sort).

Clausen's FFT on $\mathbb{S}_n$

Separation of variables

Let G be a finite group, $f\colon G \to \mathbb{C}$ and $H < G$.

- For each $g \in G/H$, define $f_g(x) = f(gx)$.
- Compute each sub-FT $\{\widehat{f}_g(\rho')\}_{\rho' \in \mathcal{R}_H}$.
- Assemble $\{\widehat{f}_g(\rho')\}_{\rho' \in \mathcal{R}_G}$.
- $$\widehat{f}(\rho) = \sum_{g \in G/H} \rho(g)\, \widehat{f}_g(\rho) \qquad \rho \in \mathcal{R}_G.$$

Clausen's FFT (1989)

In a representation adapted to $\mathbb{S}_n > \ldots > \mathbb{S}_1$, define $f_i(\tau) = f(\llbracket i, n \rrbracket \tau)$.

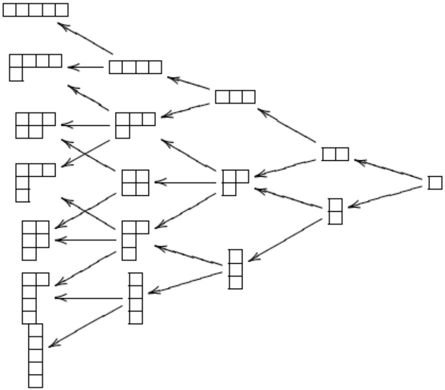
$$\widehat{f}(\rho) = \sum_{\sigma \in \mathbb{S}_n} f(\sigma)\, \rho_\lambda(\sigma) = \sum_{i=1}^{n} \sum_{\tau \in \mathbb{S}_{n-1}} f(\llbracket i, n \rrbracket \tau)\, \rho_\lambda(\llbracket i, n \rrbracket \tau) =$$

$$\sum_{i=1}^{n} \rho_\lambda(\llbracket i, n \rrbracket) \sum_{\tau \in \mathbb{S}_{n-1}} f_i(\tau)\, \rho_\lambda(\tau) = \sum_{i=1}^{n} \rho_\lambda(\llbracket i, n \rrbracket) \bigoplus_{\lambda' \in \lambda\downarrow_{n-1}} \sum_{\tau \in \mathbb{S}_{n-1}} f_i(\tau)\, \rho_{\lambda'}(\tau) =$$

$$= \sum_{i=1}^{n} \rho_\lambda(\llbracket i, n \rrbracket) \bigoplus_{\lambda' \in \lambda\downarrow_{n-1}} \widehat{f_i}(\lambda')$$

where $\lambda\downarrow_{n-1} := \{\, \lambda' \vdash n-1 \mid \lambda' \leq \lambda \,\}$.

The Bratelli diagram



Complexity

- The multiplication $\rho_\lambda(\sigma) \cdot M$ takes only $2d_\lambda^2$ operations.
- Therefore, computing $\rho_\lambda([\![i,k]\!]) \bigoplus_{\lambda'} \widehat{f}_i(\rho_{\lambda'})$ takes $2(k-i)d_\lambda^2$ ops.
- $\prod_{\lambda k} d_\lambda^2 = k!$, but at level k need $n!/k!$ FT's.

Total:

$$2n! \sum_{k=1}^{n} \sum_{i=1}^{k} (k-i) = 2n! \sum_{k=1}^{n} \frac{k(k-1)}{2} = \frac{(n+1)\,n\,(n-1)}{3}\, n!\,,$$

- Complexity of inverse transform the same (by unitarity of each level).
- May be possible to improve [Maslen, 1998][Maslen & Rockmore, 2000]

$S_n$ob

A C++ library for fast Fourier transforms on the symmetric group.

author: Risi Kondor, Columbia University (risi@caltech.edu)

Development version as of August 23, 2006 (unstable!):

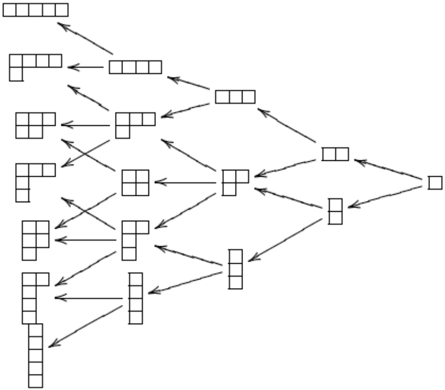
Documentation: [ps][pdf]
C++ source code: [directory]
BiBTeX entry: [bib]
Entire package: [tar.gz]

References:

1. Michael Clausen: **Fast generalized Fourier transforms.** Theoretical Computer Science **67(1)**: 55-63, 1989.
2. David K. Maslen and Daniel N. Rockmore: **Generalized FFTs --- a survey of some recent results.** Proceedings of the DIMACS Workshop on Groups and Computation, 1997. [ps]
3. K.-L. Kueh, T. Olson, D. Rockmore and K.-S. Tan: **Nonlinear approximation theory on finite groups.** Technical Report PMA-TR99-191, Department of Mathematics, Dartmouth College, 1999. [ps] [pdf]

FFT on homogeneous spaces

Recall: if $f : G/H \to \mathbb{C}$, then $\widehat{f}(\rho) := \widehat{(f\uparrow^G)}(\rho)$ where $f\uparrow^G (g) := f(gx_0)$. This gives rise to a column sparse structure in the $\widehat{f}(\rho)$ matrices determined by the multiplicity of the trivial irrep in $\rho\downarrow_H$.



The columns in $\widehat{f}(\rho_\lambda)$ are indexed by the *paths* from $\lambda_1 = (1)$ to λ in the Bratelli diagram.

Example: $\mathbb{S}_n/\mathbb{S}_{n-2}$

Proposition. The Fourier transform of a function on $\mathbb{S}_n/\mathbb{S}_{n-2}$ has the following structure:

- The one dimensional matrix $\rho_{(n)}$
- Two columns in $\rho_{(n-1,1)}$
- One column in $\rho_{(n-2,2)}$
- One column in $\rho_{(n-2,1,1)}$.