Intro to harmonic analysis on groups

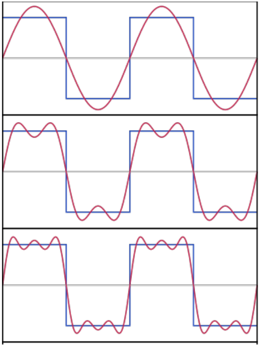
Risi Kondor

The Fourier series (1807)

Any (sufficiently smooth) function f on the unit circle (equivalently, any 2π–periodic f) can be decomposed into a sum of sinusoidal waves

$$f(x) = \sum_{k=-\infty}^{\infty} c_n \, e^{ikx} \qquad c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) \, e^{-ikx} \, dx.$$

- Workhorse of much of applied mathematics.
- Exact conditions under which it works get messy. E.g., for $f \in L_2([0, 2\pi))$ almost everywhere convergence proved only in 1966 (Carleson).



The Fourier transform

$$f(x) = \int \widehat{f}(k)\, e^{2\pi i k x}\, dk \qquad \widehat{f}(k) = \int f(x)\, e^{-2\pi i k x}\, dx$$

- Duality between time domain and Fourier domain (wave/particle duality in quantum mechanics)
- Heisenberg uncertainty principle
- Easily generalizes to $\mathbb{R}^p$.

The discrete Fourier transform (DFT)

$$f(x) = \sum_{k=0}^{n-1} \widehat{f}(k)\, e^{2\pi ikx/n} \qquad \widehat{f}(k) = \frac{1}{n} \sum_{x=0}^{n-1} f(x)\, e^{-2\pi ikx/n}$$

- Unitary transform $\mathbb{C}^n \to \mathbb{C}^n$ (with appropriate normalization).
- Can be seen as discretized version of Fourier series, or as the Fourier transform on $\{0, 1, 2, \ldots, n-1\}$.
- Foundation of all of digital signal processing.
- Fast Fourier transforms reduce computation time from $O(n^2)$ to $O(n \log n)$ [Cooley & Tukey, 1965].

Underlying principles

1. Analytic

Take a measurable space X, a space of functions on X, say $L_2(X)$, and a self-adjoint smoothing operator Υ. For example, on $X = \mathbb{R}^p$, Υ may be the time t diffusion operator

$$(\Upsilon f)(x) = \frac{1}{\sqrt{4\pi t}} \int f(y)\, e^{-\|\, x-y \,\|^2/(4t)} dy.$$

Question: How does Υ filter $L_2(X)$ into a nested sequence of spaces

$$W_\Omega = \{\, f \in L_2(X) \mid |\langle f, \Upsilon f\rangle \,/\, \langle f, f\rangle| \le \Omega \,\} \ ?$$

2. Algebraic

Now let a group G act on X inducing linear operators $T_g \colon L_2(X) \to L_2(X)$.
E.g., on on $X = \mathbb{R}^p$,

$$(T_g f)(x) = f(x - g) \qquad g \in \mathbb{R}^p.$$

Question: What are the smallest spaces fixed by these operators,

$$T_g(V) = V \qquad \forall g \in G \ ?$$

On $\mathbb{R}^p$ we are lucky because these two notions match up:

- The diffusion operator is $e^{t\nabla^2}$, where ∇^2 is the Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_p^2}.$$

- The $e^{2\pi i k \cdot x}$ Fourier basis functions are eigenfunctions of both Δ and T_g:
  - $\nabla^2 e^{2\pi i k \cdot x} = -4\pi^2 \|k\|^2 \, e^{2\pi i k \cdot x}$,
  - $T_g \, e^{2\pi i k \cdot x} = e^{2\pi i k \cdot g} \, e^{2\pi i k \cdot x}$.
- Therefore
  - $W_\Omega = \{\, f \mid \widehat{f}(k) = 0 \text{ if } \|k\|^2 \geq \Omega \,\}$ (band-limited functions)
  - $V_\kappa = \{\, f \mid \widehat{f}(k) = 0 \text{ if } k \neq \kappa \,\}$ (isotypics)

Question: Does this correspondence hold more generally?

Fourier analysis on graphs

On a finite graph $\mathcal{G}$, the analog of Δ is the graph Laplacian

$$[L]_{i,j} = \begin{cases} 1 & i \sim j \\ -d_i & i = j \\ 0 & \text{otherwise.} \end{cases}$$

It does lead to a natural measure of smoothness:

$$f^\top L f = -\sum_{i \sim j} (f_i - f_j)^2.$$

Analyzing functions in terms of the eigenfunctions of L is called **spectral graph theory**.

However (in general) on graphs there is *no* analog of translation.

More properties of the FT on $\mathbb{R}$

- The Fourier transform is
  - Linear
  - Invertible
  - $\int f(x)\,g(x)^*\,dx = \int \widehat{f}(k)\widehat{g}(k)^* dk$ (Parseval thm)

Therefore, it is essentially a unitary change of basis.

- Diagonalizes the derivative operator:

$$g(x) = \frac{d}{dx} f(x) \quad \Longrightarrow \quad \widehat{g}(k) = 2\pi i k \, \widehat{f}(k).$$

- Diagonalizes the Laplacian:

$$g(x) = \frac{d^2}{dx^2} f(x) \quad \Longrightarrow \quad \widehat{g}(k) = -4\pi k^2 \, \widehat{f}(k).$$

- Translation theorem:

$$g(x) = f(x - t) \quad \Longrightarrow \quad \widehat{g}(k) = e^{-2\pi i k t}\, \widehat{f}(k)$$

- Scaling theorem:

$$g(x) = f(\lambda x) \quad \Longrightarrow \quad \widehat{f}'(k) = |\lambda|^{-1}\, \widehat{f}(k/\lambda)$$

- Convolution theorem:

$$(f * g)(x) = \int f(x-y)\,g(y)\,dy \implies \widehat{f * g}(k) = \widehat{f}(k) \cdot \widehat{g}(k)$$

- Cross-correlation theorem:

$$(f \star g)(x) = \int f(y)^*\,g(x+y)\,dy \implies \widehat{f \star g}(k) = \widehat{f}(k)^* \cdot \widehat{g}(k)$$

  - Autocorrelation:

$$h(x) = \int f(y)^*\,f(x+y)\,dy \implies \widehat{h}(k) = \|\widehat{f}(k)\|^2$$

Fourier analysis on compact groups

Fourier tranform on $\mathbb{R}$

$$\widehat{f}(k) = \int f(x)\, e^{-2\pi i k x}\, dx$$

Observation: $\chi_k(x) = e^{-2\pi i k x}$ are exactly the characters of $\mathbb{R}$.

Locally Compact Abelian groups

The Fourier transform of a function on an LCA group G with Haar measure μ is

$$\widehat{f}(\chi) = \int_G f(x)\,\chi(x)\,d\mu \qquad \chi \in \widehat{G}.$$

- The dual object is itself a group: $T \leftrightarrow \mathbb{Z}$, $\mathbb{R} \leftrightarrow \mathbb{R}$, and for finite groups $\widehat{G} \cong G$ (Pontryagin duality).
- This covers the Fourier series and the Fourier transform.

Compact non-Abelian groups

The Fourier transform of a function on a compact group G with Haar measure μ is

$$\widehat{f}(\rho) = \int_G f(x)\,\rho(x)\,d\mu(x) \qquad \rho \in \mathcal{R},$$

where $\mathcal{R}$ is a complete set of inequivalent irreducible representations (irreps).

- Now the dual object is no longer a group, but a set of representations (Tannaka–Krein duality).
- If G is finite, $\mathcal{R}$ is finite. If G is compact, $\mathcal{R}$ is countable.
- Each Fourier component $\widehat{f}(\rho)$ is a *matrix*.

In the following, we will always assume that each ρ is unitary. Every representation is over $\mathbb{C}$.

Properties

Invertibility

Forward transform:

$$\widehat{f}(\rho) = \int_G f(x)\,\rho(x)\,d\mu(x) \qquad\qquad \rho \in \mathcal{R}.$$

Inverse transform:

$$f(x) = \frac{1}{\mu(G)} \sum_{\rho \in \mathcal{R}} d_\rho \operatorname{tr}\left[\widehat{f}(\rho)\,\rho(x^{-1})\right] \qquad\qquad x \in G.$$

- Just as before (with respect to the appropriate scaled matrix norms), this transform is unitary.
- The $e^{\rho}_{i,j}(x) = \sqrt{d_\rho}\,[\rho(x)]_{i,j}$ functions form an orthogonormal basis (Peter-Weyl theorem).

Left-translation

Theorem. Given $f\colon G \to \mathbb{C}$ and $t \in G$, define $f^t(x) = f(t^{-1}x)$. Then

$$\widehat{f^t}(\rho) = \rho(t) \cdot \widehat{f}(\rho) \qquad \rho \in \mathcal{R}.$$

Proof.

$$\int f^t(x)\,\rho(x)\,d\mu(x) = \int f(t^{-1}x)\,\rho(x)\,d\mu(x) =$$

$$\int f(x)\,\rho(tx)\,d\mu(x) = \int f(x)\,\rho(t)\,\rho(x)\,d\mu(x) = \rho(t)\,\widehat{f}(\rho)$$

Left-translation

- Convolution theorem:

$$(f * g)(x) = \int f(xy^{-1})\, g(y)\, d\mu(y) \implies \widehat{f * g}(\rho) = \widehat{f}(\rho) \cdot \widehat{g}(\rho)$$

- Cross-correlation theorem:

$$(f \star g)(x) = \int f(xy)\, g(y)^*\, d\mu(y) \implies \widehat{f \star g}(\rho) = \widehat{f}(\rho) \cdot \widehat{g}(\rho)^\dagger$$

Right-translation

Theorem. Given $f\colon G \to \mathbb{C}$ and $t \in G$, define $f^{(t)}(x) = f(xt^{-1})$. Then

$$\widehat{f^{(t)}}(\rho) = \widehat{f}(\rho) \cdot \rho(t) \qquad\qquad \rho \in \mathcal{R}.$$

Proof.

$$\int f^{(t)}(x)\,\rho(x)\,d\mu(x) = \int f(xt^{-1})\,\rho(x)\,d\mu(x) =$$

$$\int f(x)\,\rho(xt)\,d\mu(x) = \int f(x)\,\rho(x)\,\rho(t)\,d\mu(x) = \widehat{f}(\rho)\,\rho(t)$$

Invariant subspaces

- To left-translation:

$$W_{\rho,j} = \text{span}\{\, e^{\rho}_{i,j} \mid j = 1, \ldots, d_\rho \,\} \qquad \rho \in \mathcal{R} \qquad j = 1, \ldots, d_\rho.$$

- To right-translation:

$$W_{\rho,i} = \text{span}\{\, e^{\rho}_{i,j} \mid i = 1, \ldots, d_\rho \,\} \qquad \rho \in \mathcal{R} \qquad i = 1, \ldots, d_\rho.$$

- To left- and right-translation:

$$V_\rho = \text{span}\{\, e^{\rho}_{i,j} \mid i, j = 1, \ldots, d_\rho \,\} \qquad \rho \in \mathcal{R}.$$

The group algebra

The group algebra $\mathbb{C}[G]$ is a space with orthonormal basis $\{ e_x \mid x \in G \}$ and a notion of multipilication defined by

$$e_x e_y = e_{xy} \qquad \forall x, y \in G.$$

Letting $\langle f, e_x \rangle = f(x)$ and extending to the rest of $\mathbb{C}[G]$ by linearity, for any $f, g \in \mathbb{C}[G]$,

$$(fg)(x) = \int f(xy^{-1})\, g(y)\, d\mu(y) = (f * g)(x).$$

The group algebra of any compact group is semi-simple, i.e., it decomposes into a direct sum of simple algebras.

The group algebra

- The group algebra decomposes into a sum of simple algebras:

$$\mathbb{C}[G] = \bigoplus_{\rho} V_{\rho}. \tag{1}$$

  Each V_ρ is called an **isotypic**, and corresponds to a single Fourier matrix $\widehat{f}(\rho)$. This decomposition is unique.

- Each V_ρ further decomposes into a sum of d_ρ left G–modules

$$V_\rho = W_1^\rho \oplus W_2^\rho \oplus \ldots \oplus W_{d_\rho}^\rho \tag{2}$$

  corresponding to each column of $\widehat{f}(\rho)$. This decomposition is not unique, i.e., it depends on the choice of $\mathcal{R}$.

The Fourier transform is a projection of f onto a basis adapted to (1) and (2).

Fourier analysis on homogeneous spaces

Group actions

- So far we have considered:
  - f is a function on a compact group G.
  - G acts on G by $t\colon x \mapsto tx$ inducing $f \mapsto f^t$, where $f^t(x) = f(t^{-1}x)$ (similarly for the right-action and right-translation).
- In practice it is often more common that:
  - f is a function on a set X.
  - G acts on X transitively by $t\colon x \mapsto tx$, inducing $f \mapsto f^t$, where $f^t(x) = f(t^{-1}x)$.

Example: The rotation group $\mathrm{SO}(3)$ and the sphere S^2. The symmetric group acting on a matrix by permuting rows/columns.

Homogeneous spaces

Assume that G acts on X transitively by $x \mapsto gx$.

- Pick some $x_0 \in X$.
- The subset of G fixing x_0 is a subgroup H of G.
- Each set $gH = \{ gh \mid h \in H \}$ is called a left H–coset.
- The set of left H–cosets we denote G/H.
- $\{ gH \mid gH \in G/H \}$ forms a partition of G.
- $yx_0 = y'x_0$ if and only if y, y' belong to the same coset.
- Therefore, we have a bijection

$$X \leftrightarrow G/H.$$

X is called a **homogeneous space** of G.

Example: $S^2 \sim \mathrm{SO}(3)/\mathrm{SO}(2).$

FT on homogeneous spaces

Now $L(X)$ is only a G-module, not a $\mathbb{C}[G]$ algebra. However, we can still ask, how it decomposes into a sum of invariant modules.

Definition. The Fourier transform of $f\colon X \to \mathbb{C}$ is the FT of the induced function $f{\uparrow}^G(g) = f(gx_0)$, i.e.,

$$\widehat{f}(\rho) = \int_G f(gx_0)\,\rho(g)\,d\mu(g) \qquad\qquad \rho \in \mathcal{R}$$

Adapted bases

We say that the repesentation ρ of G is adapted to $H \leq G$, if $\rho{\downarrow}_H$ is of the block diagonal form

$$\rho{\downarrow}_H(h) = \bigoplus_{\rho' \in R_\rho} \rho'(h) \qquad h \in H \tag{3}$$

for some multiset R_ρ of irreps of H.

We use $m_{\mathrm{tr}}(\rho)$ to denote the multiplicity of the trivial representation in the decomposition (3).

FT on homogeneous spaces

Theorem. If $f\colon X \to \mathbb{C}$ and $\widehat{f}$ is expressed in a basis adapted to $H \leq G$, then $\widehat{f}(\rho)$ has at most $m_{\mathrm{tr}}(\rho)$ non-zero columns.

Proof.

$$\widehat{f}(\rho) = \int_{g\in G/H} \int_{h\in H} f(gx_0)\, \rho(gh)\, d\mu(g)\, d\mu(h) =$$

$$\left[\int_{g\in G/H} f(gx_0)\, \rho(g)\, d\mu(g)\right] \underbrace{\left[\int_{h\in H} \rho(h)\, d\mu(h)\right]}$$

Proof (continued)

$$\int_{h\in H} \rho(h)\, d\mu(h) = \bigoplus_{\rho' \in R} \int_{h\in H} \rho'(h)\, d\mu(h)$$

If ρ' is the trivial irrep of H, then $\int_{h\in H} \rho'(h)\, d\mu(h) = \mu(H)$. However, by orthogonality of the Fourier basis functions, for any other irrep, $\int_{h\in H} \rho'(h)\, d\mu(h) = 0$.

Example

The rotation group $\mathrm{SO}(3)$ is parametrized by the Euler angles (θ, ϕ, ψ), and the irreps are given by the $D^{(0)}, D^{(1)}, D^{(2)}, \ldots$ Wigner matrices, where

$$[D^{(\ell)}]_{m,m'} = e^{-im'\psi}\, Y_\ell^m(\theta, \phi), \qquad m, m' = -\ell, \ldots, \ell$$

where

$$Y_\ell^m(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}}\; P_l^m(\cos\theta)\, e^{im\phi},$$

are the spherical harmonics. Clearly, the Wigner matrices are adapted to the subgroup $\mathrm{SO}(2)$ that rotates ψ. In particular,

$$D^{(\ell)}\downarrow_{\mathrm{SO}(2)}(\psi) = \bigoplus_{m'=-\ell}^{\ell} \chi_{m'}(\psi) \qquad \chi_{m'}(\psi) = e^{-im'\psi},$$

so the multiplicity of the trivial irrep χ_0 is 1.

Example

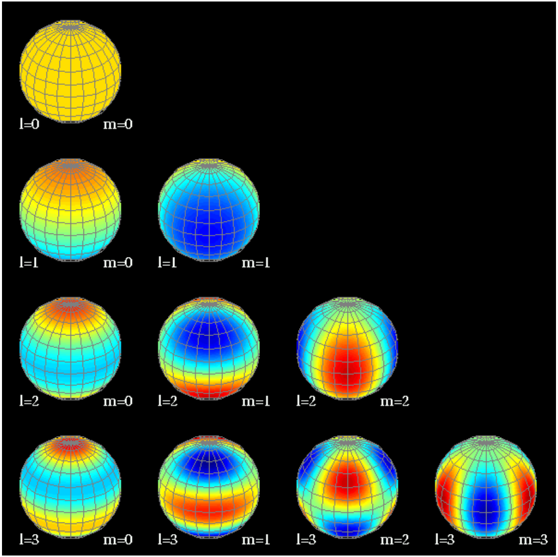
The Fourier transform of $f\colon S^2 \to \mathbb{C}$ is

$$\widehat{f}(\ell) = \int_{\mathrm{SO}(3)} f(Rx_0)\, D^{(\ell)}(R)\, d\mu(R) \qquad \ell = 0, 1, 2, \ldots,$$

but by our theorem only the middle column of each of these matrices is non-zero, which yields exactly the celebrated spherical harmonic expansion

$$f(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{m} f_\ell^m \, Y_\ell^m(\theta, \phi).$$

Example



APPLICATIONS

- Invariants to group actions:
  - Computer vision
  - Graph invariants
- Band–limited approximations on $\mathbb{S}_n$:
  - Multi-object tracking
- Wavelets on $\mathbb{S}_n$
- Learning on $\mathbb{S}_n$:
  - Ranking problems
- Optimization on $\mathbb{S}_n$:
  - Fast QAP solvers.