# The Geometric Weil Representation and Pseudo-Random Vectors

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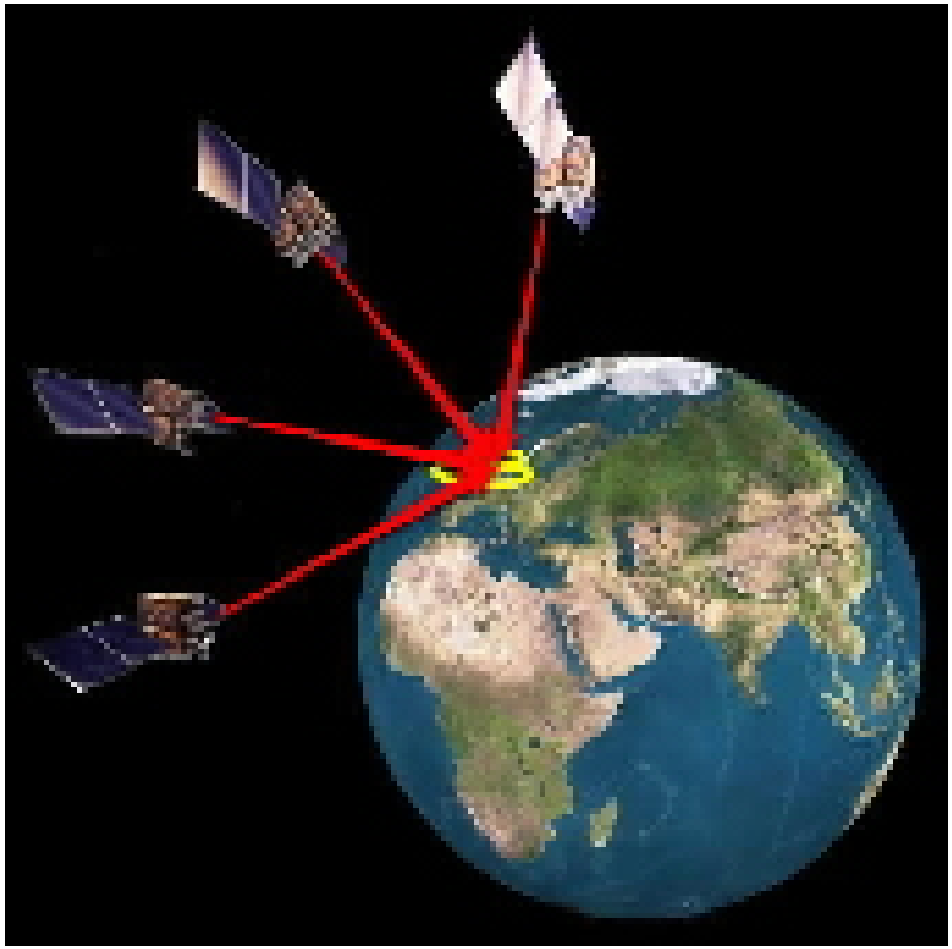
Madison

August 5, 2014

# (0) MOTIVATION - GPS



**CLIENT WANT: Coordinates of satellite and time delay (enables to calculate distance to a satellite)?**

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## Fact

*Client receives*

$$R[n] = b \cdot \alpha_0 \cdot \psi[\omega_0 n] \cdot S[n - \tau_0] + \mathcal{W}[n], \quad n \in \mathbb{F}_p,$$

$\alpha_0 \in \mathbb{C}$ *attenuation,* $\omega_0 \in \mathbb{F}_p$ *Doppler,* $\tau_0 \in \mathbb{F}_p$ *delay,* $\mathcal{W} \in \mathcal{H}$ *random white noise.*

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## Problem (The GPS Problem)

*Design* $S \in \mathcal{H}$*, and effective method to extract* $(b, \tau_0)$*, using* $R$ *and* $S$*.*

# SOLUTION - Matched Filter

## Definition

Matched filter

$$\begin{cases} \mathcal{M}(R,S): \overbrace{\mathbb{F}_p \times \mathbb{F}_p}^{\text{Time-Frequency}} \to \mathbb{C}, \\ \mathcal{M}(R,S)[\tau,\omega] = \langle R[n] \ , \ \psi[\omega n] \cdot S[n-\tau] \rangle \ . \end{cases}$$

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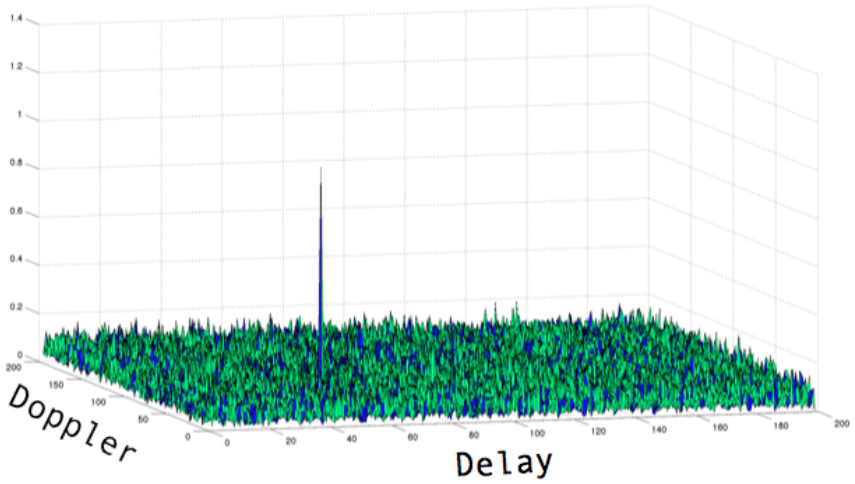
- **Question:** What $S$ to use for extracting $(\tau_0, \omega_0)$ from $\mathcal{M}(R,S)$ ?

# Solution - MATCHED FILTER

- Typical solution: $S$ = pseudo-random (PR)

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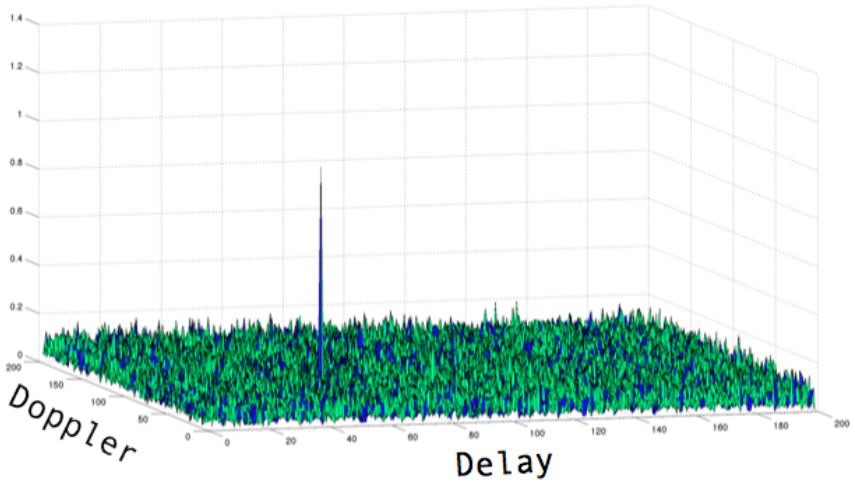


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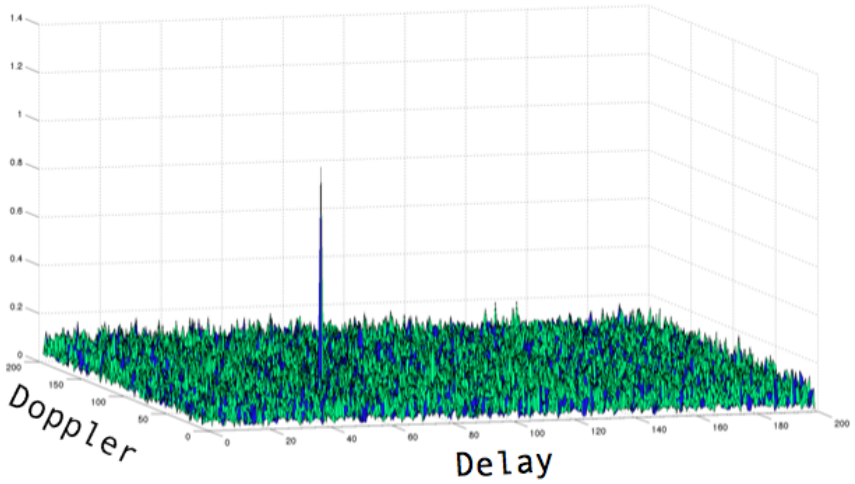


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- **Task:** Find method to construct explicit PR sequences.

# WEIL REP'N SEQUENCES - Idea

- Weil rep'n (**Weil** 64)

$$\begin{cases} \rho : SL_2(\mathbb{F}_p) \to GL(\mathbb{C}(\mathbb{F}_p)), \\ \rho \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = DFT. \end{cases}$$

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  - $\dim \mathcal{H}_\chi = 1, \quad \varphi_\chi \in \mathcal{H}_\chi, \quad \left\| \varphi_\chi \right\| = 1.$

# Weil Rep'n Sequences - IDEA

## Theorem (Pseudo Randomness)

*For* $(\tau, \omega) \neq (0, 0)$

$$\left|\mathcal{M}(\varphi_\chi\ ,\ \varphi_\chi)[\tau, \omega]\right| \leq \frac{2}{\sqrt{p}}.$$

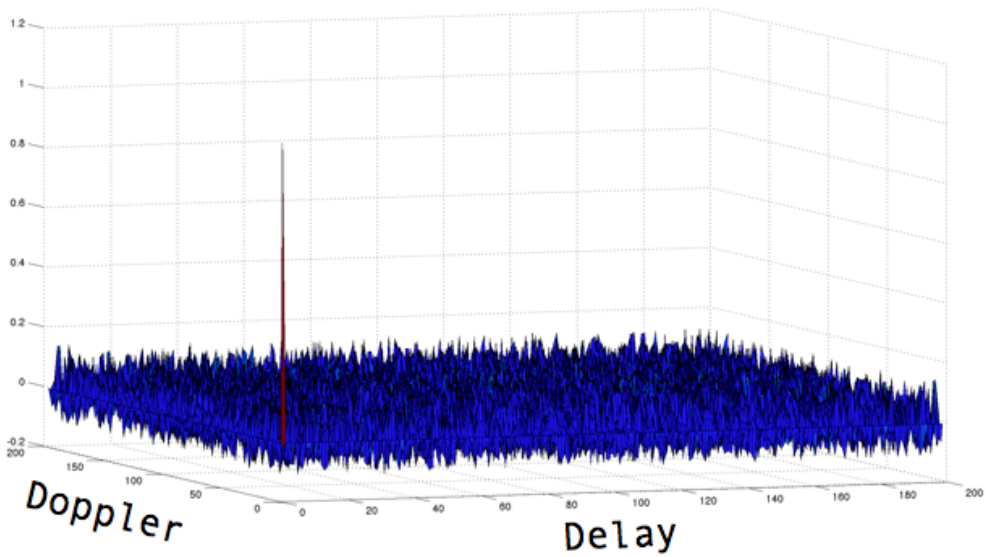


Figure: $\mathcal{M}(\varphi_\chi, \varphi_\chi)$, $q = 199$.

# BOUNDS VIA GEOMETRIC WEIL REP'N - Idea

- $\ell$-adic sheaves (**Grothendieck** 60s)

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- Application: Geometrizing $\mathcal{M}(\varphi_\chi, \varphi_\chi)$ and obtain the bound.

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**Theorem (Stone–von Neumann)**

*There exists a unique (up to $\simeq$) irreducible representation* $\underbrace{\pi : H \to GL(\mathcal{H})}_{\text{Heisenberg rep'n}}$ *s.t.* $\pi(z) = \psi(z) \cdot Id_{\mathcal{H}}$, $z \in Z(H) = k$.

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  $$\pi \overset{\rho(g)}{\widetilde{\longrightarrow}} \pi^g(v, z) = \pi(gv, z),$$

  i.e.,

  $$\rho(g)\pi(h)\rho(g)^{-1} = \pi(g[h]), \quad \text{for every } h \in H. \tag{1}$$

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*Formula for $\rho$ ?*

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- *Proof.* Geometric Weil Rep'n.

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  2. $\mathbf{X} = \mathbb{G}_m$, $Fr(x) = x^q$, $\mathbb{G}_m(k) = k^*$, $\chi : k^* \to \mathbb{C}^*$ – multiplicative character,

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1. *Multiplicativity. Canonical isomorphism* $\theta : \mathcal{K}(g_1, \cdot) * \mathcal{K}(g_2, \cdot) \widetilde{\to} \mathcal{K}(g_1 g_2, \cdot)$.

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$$\mathcal{K}(g, v) = \mathcal{L}_{\sigma[(-1)^n \det(g-I))]} \otimes \mathcal{L}_{\psi[\Omega(\frac{g+I}{g-I} v, v)]}[2n](n).$$

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**Theorem (Pseudo-randomness)**

*For $v \neq 0$ we have*

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$\mathbf{X}/\mathbb{F}_q$, $\dim \mathbf{X} = 1$, $(\mathcal{F}, \bigtriangledown)$ *Weil line bundle with flat connection and weight zero. Then*

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- In our case $\mathcal{F}$ is explicit. Can compute $c = 2$. Done!

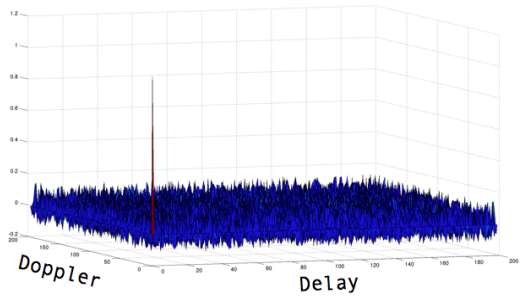


Figure: $\left\langle \varphi_\chi,\ \pi(\tau, \omega)\varphi_\chi \right\rangle$, $q = 199$.

# THANK YOU

Thank You!