Diagonalization of the Discrete Fourier Transform using Weil Representation

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Problem (Diagonalization)

Find natural basis of eigenfunctions for DFT.

Solution - Idea

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- **Answer:** Characterization of DFT.

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- Intertwining relations

$$\begin{cases} DFT \circ L_\tau = M_\tau \circ DFT, \\ DFT \circ M_\omega = L_{-\omega} \circ DFT. \end{cases}$$

Characterization of DFT - Cont.

- Combine

$$\begin{cases} \pi : \mathbb{F}_p \times \mathbb{F}_p \to U(\mathcal{H}), \\ \pi(\tau, \omega) = \psi(-\frac{1}{2}\tau\omega) \cdot M_\omega \circ L_\tau \end{cases}$$

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$$\Sigma_W : \ DFT \circ \pi \begin{pmatrix} \tau \\ \omega \end{pmatrix} = \pi(\overbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}^{W} \begin{pmatrix} \tau \\ \omega \end{pmatrix}) \circ DFT.$$

System of p^2 linear equations.

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- $\Longrightarrow$ *DFT* is characterized by Σ_W.

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- Note

$$W \in SL_2(\mathbb{F}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; \ a, b, c, d \in \mathbb{F}_p, \quad ad - bc = 1 \right\}.$$

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- Generalization: $g \in SL_2(\mathbb{F}_p)$

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Theorem

$\exists!$ *collection of operators* $\rho(g) \in Sol(\Sigma_g)$, $g \in SL_2(\mathbb{F}_p)$, *such that*

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- The homomorphism

$$\rho : SL_2(\mathbb{F}_p) \to U(\mathcal{H}), \quad \mathcal{H} = \mathbb{C}(\mathbb{F}_p),$$

is called the **Weil Representation.**

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Proof.

$P_W(x) = \det(xI - W) = x^2 + 1$. Hence $\lambda(W) = \pm\sqrt{-1}$. □

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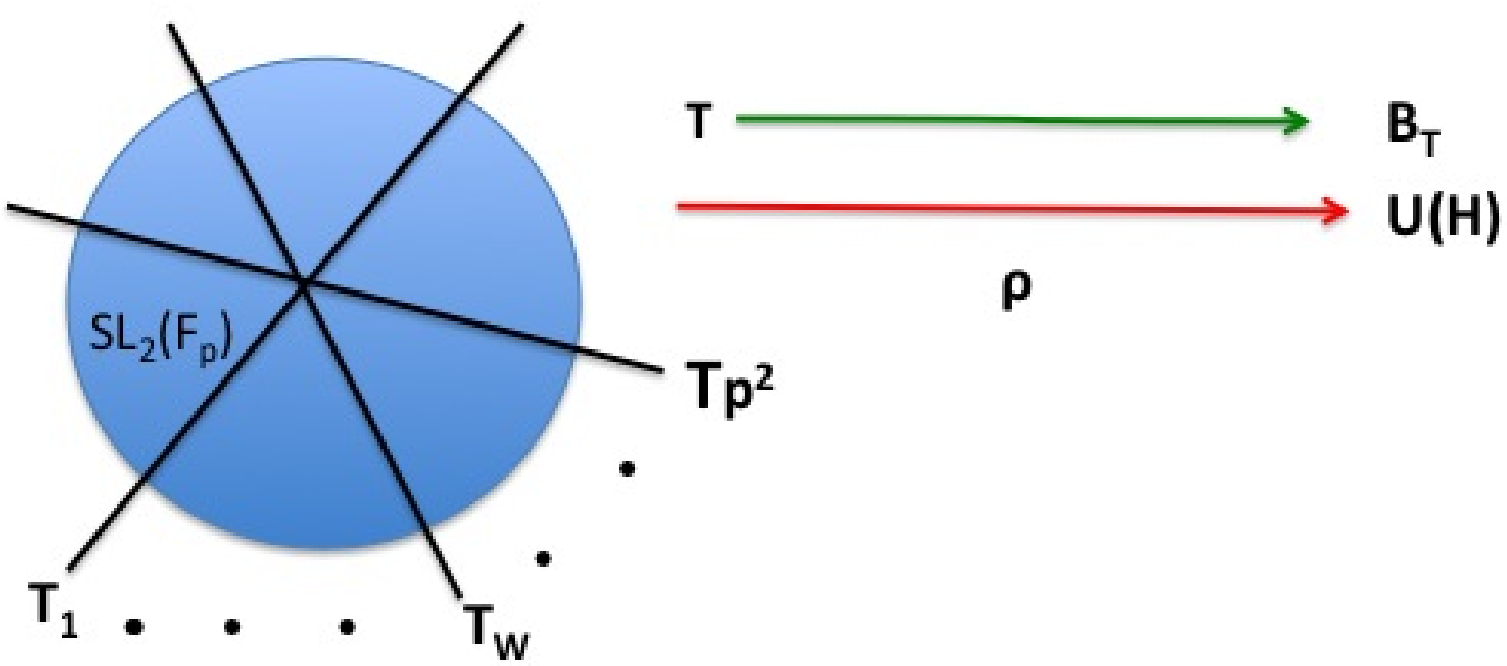
- Obtained the **Canonical basis of eigenfunctions of DFT**

$$B_{T_W} = \{\varphi_\chi \in \mathcal{H}_\chi\}.$$

(IV) The Oscillator Dictionary

- Generalization



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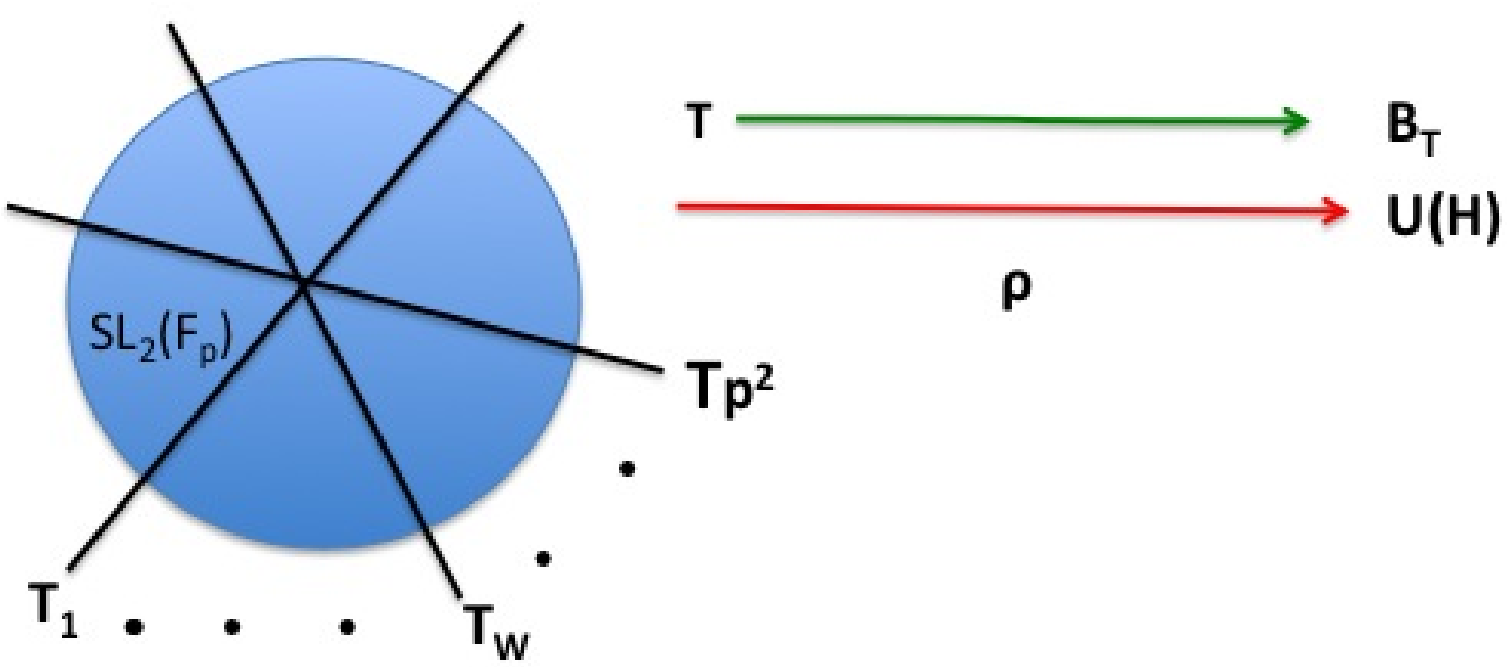

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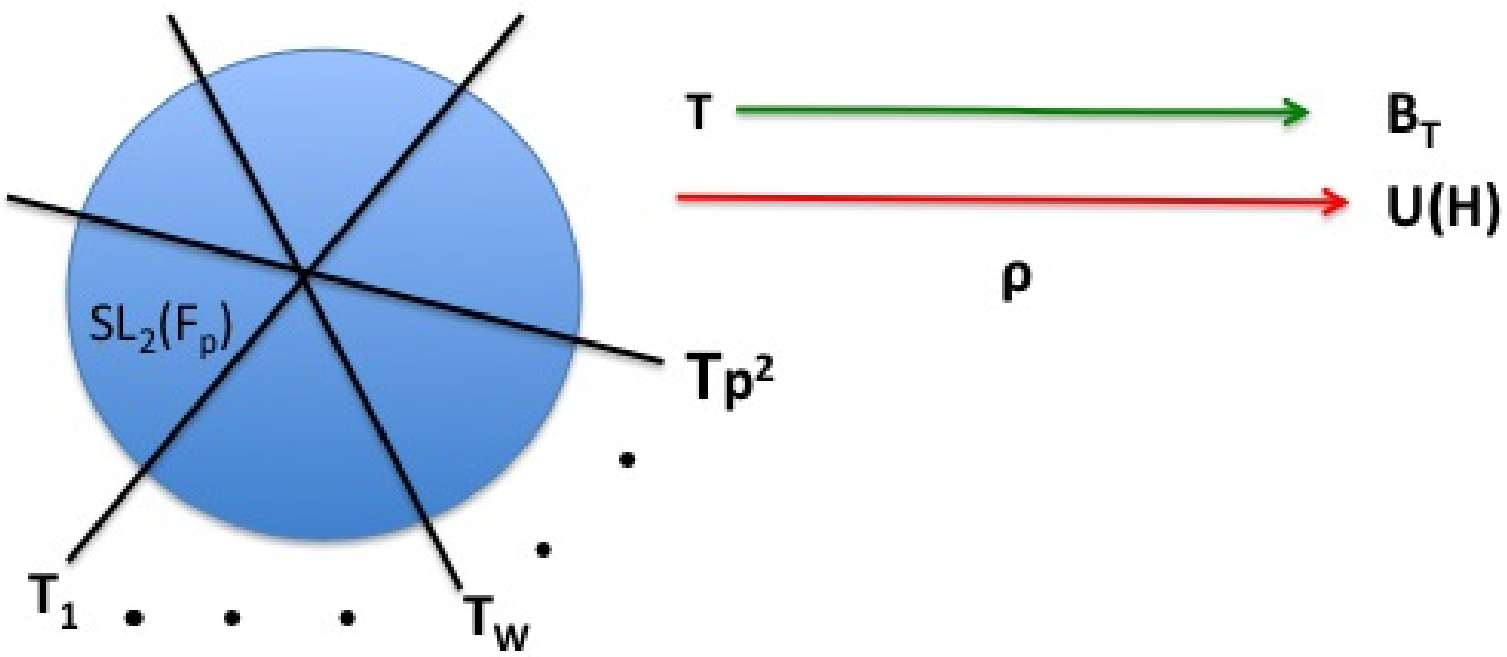

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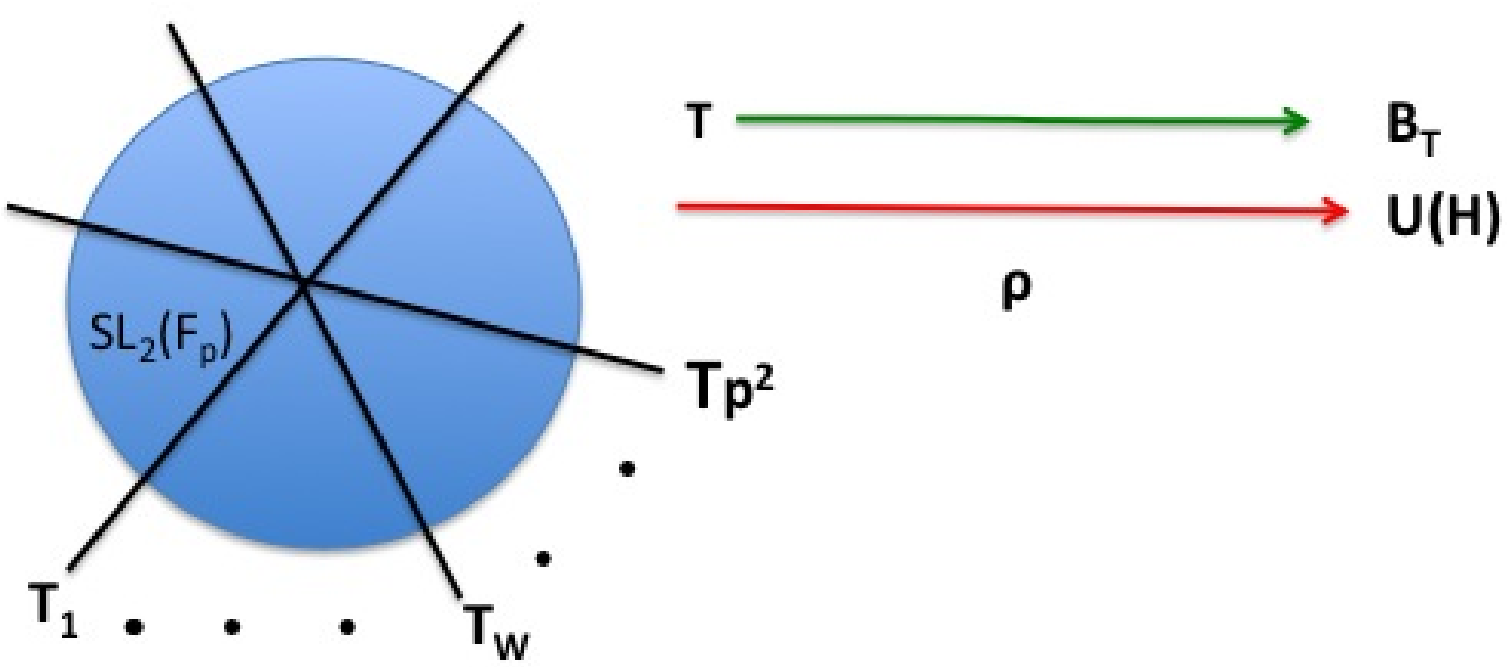

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2. **Cross-correlation.** *For every* $\varphi \neq \phi \in D$

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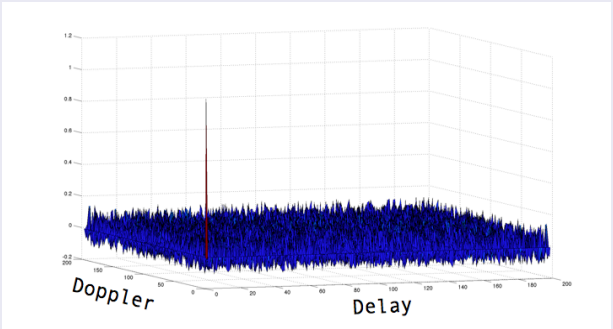

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