The Heisenberg Representation and the Fast Fourier Transform

Shamgar Gurevich

UW Madison

July 31, 2014

Motivation:

- Discrete Fourier Transform

$$DFT = \frac{1}{\sqrt{N}} \left(e^{\frac{2\pi i}{N} \tau \cdot \omega} \right)_{0 \leq \tau, \omega \leq N-1}$$

Motivation:

- Discrete Fourier Transform

$$DFT = \frac{1}{\sqrt{N}} \left(e^{\frac{2\pi i}{N} \tau \cdot \omega} \right)_{0 \leq \tau, \omega \leq N-1}$$

- Compute:

$$\widehat{f} = DFT[f]; \qquad f = \begin{pmatrix} f(0) \\ . \\ . \\ . \\ f(N-1) \end{pmatrix}; \qquad \text{Fast!!}$$

Motivation:

- Discrete Fourier Transform

$$DFT = \frac{1}{\sqrt{N}} \left(e^{\frac{2\pi i}{N} \tau \cdot \omega} \right)_{0 \leq \tau, \omega \leq N-1}$$

- Compute:

$$\widehat{f} = DFT[f]; \qquad f = \begin{pmatrix} f(0) \\ . \\ . \\ . \\ f(N-1) \end{pmatrix}; \qquad \text{Fast!!}$$

- Cooley–Tukey (1965): $O(N \cdot \log(N))$ operations!

Solution (Auslander–Tolimieri)

(I) Heisenberg Group Representation

- $\mathcal{H} = \mathbb{C}(\mathbb{Z}_N)$ — Hilbert space of digital signals.

Solution (Auslander–Tolimieri)

(I) Heisenberg Group Representation

- $\mathcal{H} = \mathbb{C}(\mathbb{Z}_N)$ — Hilbert space of digital signals.
  - $f : \{0, ..., N-1\} \to \mathbb{C}$.

Solution (Auslander–Tolimieri)

(I) Heisenberg Group Representation

- $\mathcal{H} = \mathbb{C}(\mathbb{Z}_N)$ — Hilbert space of digital signals.
  - $f : \{0, ..., N-1\} \to \mathbb{C}$.
- Basic operations

Solution (Auslander–Tolimieri)

(I) Heisenberg Group Representation

- $\mathcal{H} = \mathbb{C}(\mathbb{Z}_N)$ — Hilbert space of digital signals.
  - $f : \{0, ..., N-1\} \to \mathbb{C}$.
- Basic operations
  - Time shift: $\tau \in \mathbb{Z}_N$,

$$L_\tau : \mathcal{H} \to \mathcal{H},$$
$$L_\tau[f](t) = f(t+\tau), \ t \in \mathbb{Z}_N.$$

Solution (Auslander–Tolimieri)

(I) Heisenberg Group Representation

- $\mathcal{H} = \mathbb{C}(\mathbb{Z}_N)$ — Hilbert space of digital signals.
  - $f : \{0, ..., N-1\} \to \mathbb{C}$.
- Basic operations
  - Time shift: $\tau \in \mathbb{Z}_N$,

$$L_\tau : \mathcal{H} \to \mathcal{H},$$
$$L_\tau[f](t) = f(t+\tau), \ t \in \mathbb{Z}_N.$$

  - Frequency shift: $\omega \in \mathbb{Z}_N$,

$$M_\omega : \mathcal{H} \to \mathcal{H},$$
$$M_\omega[f](t) = e^{\frac{2\pi i}{N}\omega t} f(t).$$

Solution (Auslander–Tolimieri)

(I) Heisenberg Group Representation

- $\mathcal{H} = \mathbb{C}(\mathbb{Z}_N)$ — Hilbert space of digital signals.
  - $f : \{0, ..., N-1\} \to \mathbb{C}$.
- Basic operations
  - Time shift: $\tau \in \mathbb{Z}_N$,

$$L_\tau : \mathcal{H} \to \mathcal{H},$$
$$L_\tau[f](t) = f(t+\tau), \ t \in \mathbb{Z}_N.$$

  - Frequency shift: $\omega \in \mathbb{Z}_N$,

$$M_\omega : \mathcal{H} \to \mathcal{H},$$
$$M_\omega[f](t) = e^{\frac{2\pi i}{N}\omega t} f(t).$$

- Note:

$$M_\omega \circ L_\tau = e^{\frac{2\pi i}{N}\omega\tau} \cdot L_\tau \circ M_\omega \quad — \quad \textit{Heisenberg commutation relations}$$

Heisenberg Rep'n, cont.

- Combine: $\tau, \omega, z \in \mathbb{Z}_N$

$$\pi(\tau, \omega, z) = e^{\frac{2\pi i}{N}\{\frac{1}{2}\omega\tau + z\}} \cdot M_\omega \circ L_\tau.$$

Heisenberg Rep'n, cont.

- Combine: $\tau, \omega, z \in \mathbb{Z}_N$

$$\pi(\tau, \omega, z) = e^{\frac{2\pi i}{N}\{\frac{1}{2}\omega\tau + z\}} \cdot M_\omega \circ L_\tau.$$

- Identity:

$$\pi(\tau, \omega, z) \circ \pi(\tau', \omega', z') = \pi(\tau + \tau', \omega + \omega', z + z' + \frac{1}{2}\begin{vmatrix} \tau & \omega \\ \tau' & \omega' \end{vmatrix}).$$

Heisenberg Rep'n, cont.

- Combine: $\tau, \omega, z \in \mathbb{Z}_N$

$$\pi(\tau, \omega, z) = e^{\frac{2\pi i}{N}\{\frac{1}{2}\omega\tau + z\}} \cdot M_\omega \circ L_\tau.$$

- Identity:

$$\pi(\tau, \omega, z) \circ \pi(\tau', \omega', z') = \pi(\tau + \tau', \omega + \omega', z + z' + \frac{1}{2}\begin{vmatrix} \tau & \omega \\ \tau' & \omega' \end{vmatrix}).$$

- Question. How to think on this?

Heisenberg Rep'n, cont.

- Combine: $\tau, \omega, z \in \mathbb{Z}_N$

$$\pi(\tau, \omega, z) = e^{\frac{2\pi i}{N}\{\frac{1}{2}\omega\tau + z\}} \cdot M_\omega \circ L_\tau.$$

- Identity:

$$\pi(\tau, \omega, z) \circ \pi(\tau', \omega', z') = \pi(\tau + \tau', \omega + \omega', z + z' + \frac{1}{2}\begin{vmatrix} \tau & \omega \\ \tau' & \omega' \end{vmatrix}).$$

- Question. How to think on this?
- Answer. Heisenberg group:

Heisenberg Rep'n, cont.

- Combine: $\tau, \omega, z \in \mathbb{Z}_N$

$$\pi(\tau, \omega, z) = e^{\frac{2\pi i}{N}\{\frac{1}{2}\omega\tau + z\}} \cdot M_\omega \circ L_\tau.$$

- Identity:

$$\pi(\tau, \omega, z) \circ \pi(\tau', \omega', z') = \pi(\tau + \tau', \omega + \omega', z + z' + \frac{1}{2}\begin{vmatrix} \tau & \omega \\ \tau' & \omega' \end{vmatrix}).$$

- Question. How to think on this?
- Answer. Heisenberg group:
  - $H = \underbrace{\mathbb{Z}_N \times \mathbb{Z}_N}_{V} \times \underbrace{\mathbb{Z}_N}_{Z}$;

Heisenberg Rep'n, cont.

- Combine: $\tau, \omega, z \in \mathbb{Z}_N$

$$\pi(\tau, \omega, z) = e^{\frac{2\pi i}{N}\{\frac{1}{2}\omega\tau + z\}} \cdot M_\omega \circ L_\tau.$$

- Identity:

$$\pi(\tau, \omega, z) \circ \pi(\tau', \omega', z') = \pi(\tau + \tau', \omega + \omega', z + z' + \frac{1}{2}\begin{vmatrix} \tau & \omega \\ \tau' & \omega' \end{vmatrix}).$$

- Question. How to think on this?
- Answer. Heisenberg group:
  - $H = \underbrace{\mathbb{Z}_N \times \mathbb{Z}_N}_{V} \times \underbrace{\mathbb{Z}_N}_{Z}$;
  - $(v, z) \cdot (v', z') = (v + v', z + z' + \frac{1}{2}\begin{vmatrix} v \\ v' \end{vmatrix})$;

Heisenberg Rep'n, cont.

- Combine: $\tau, \omega, z \in \mathbb{Z}_N$

$$\pi(\tau, \omega, z) = e^{\frac{2\pi i}{N}\{\frac{1}{2}\omega\tau + z\}} \cdot M_\omega \circ L_\tau.$$

- Identity:

$$\pi(\tau, \omega, z) \circ \pi(\tau', \omega', z') = \pi(\tau + \tau', \omega + \omega', z + z' + \frac{1}{2}\begin{vmatrix} \tau & \omega \\ \tau' & \omega' \end{vmatrix}).$$

- Question. How to think on this?
- Answer. Heisenberg group:
  - $H = \underbrace{\mathbb{Z}_N \times \mathbb{Z}_N}_{V} \times \underbrace{\mathbb{Z}_N}_{Z}$;
  - $(v, z) \cdot (v', z') = (v + v', z + z' + \frac{1}{2}\begin{vmatrix} v \\ v' \end{vmatrix})$;
  - $(0, 0) \cdot (v, z) = (v, z) \cdot (0, 0) = (v, z)$;

Heisenberg Rep'n, cont.

- Combine: $\tau, \omega, z \in \mathbb{Z}_N$

$$\pi(\tau, \omega, z) = e^{\frac{2\pi i}{N}\{\frac{1}{2}\omega\tau + z\}} \cdot M_\omega \circ L_\tau.$$

- Identity:

$$\pi(\tau, \omega, z) \circ \pi(\tau', \omega', z') = \pi(\tau + \tau', \omega + \omega', z + z' + \frac{1}{2}\begin{vmatrix} \tau & \omega \\ \tau' & \omega' \end{vmatrix}).$$

- Question. How to think on this?
- Answer. Heisenberg group:
  - $H = \underbrace{\mathbb{Z}_N \times \mathbb{Z}_N}_{V} \times \underbrace{\mathbb{Z}_N}_{Z}$;
  - $(v, z) \cdot (v', z') = (v + v', z + z' + \frac{1}{2}\begin{vmatrix} v \\ v' \end{vmatrix})$;
  - $(0, 0) \cdot (v, z) = (v, z) \cdot (0, 0) = (v, z)$;
  - $(v, z) \cdot (-v, -z) = (-v, -z) \cdot (v, z) = (0, 0)$.

Heisenberg Rep'n, cont.

- Summary: Heisenberg Rep'n

$$\begin{cases} \pi : H \longrightarrow GL(\mathcal{H}); \\ \pi(\tau, \omega, z) = e^{\frac{2\pi i}{N}\{\frac{1}{2}\omega\tau + z\}} \cdot M_\omega \circ L_\tau; \\ \pi(h \cdot h') = \pi(h) \circ \pi(h') \text{ — homomorphism.} \end{cases}$$

Heisenberg Rep'n, cont.

- Summary: Heisenberg Rep'n

$$\begin{cases} \pi : H \to GL(\mathcal{H}); \\ \pi(\tau, \omega, z) = e^{\frac{2\pi i}{N}\{\frac{1}{2}\omega\tau + z\}} \cdot M_\omega \circ L_\tau; \\ \pi(h \cdot h') = \pi(h) \circ \pi(h') \text{ — homomorphism.} \end{cases}$$

Definition

A representation of a group H on a complex vector space $\mathcal{H}$ is a homomorphism

$$\begin{aligned} \pi &: H \to GL(\mathcal{H}), \\ \pi(h \cdot h') &= \pi(h) \circ \pi(h'). \end{aligned}$$

Heisenberg Rep'n, cont.

- Summary: Heisenberg Rep'n

$$\begin{cases} \pi : H \longrightarrow GL(\mathcal{H}); \\ \pi(\tau, \omega, z) = e^{\frac{2\pi i}{N}\{\frac{1}{2}\omega\tau + z\}} \cdot M_\omega \circ L_\tau; \\ \pi(h \cdot h') = \pi(h) \circ \pi(h') \text{ — homomorphism.} \end{cases}$$

Definition

A representation of a group H on a complex vector space $\mathcal{H}$ is a homomorphism

$$\begin{aligned} \pi &: H \longrightarrow GL(\mathcal{H}), \\ \pi(h \cdot h') &= \pi(h) \circ \pi(h'). \end{aligned}$$

- Question. DFT ?

(II) Representation Theory

Definitions

We say that $(\pi_1, H, \mathcal{H}_1)$, $(\pi_2, H, \mathcal{H}_2)$ are equivalent, $\pi_1 \simeq \pi_2$, if

$$\exists\ \alpha : \mathcal{H}_1 \widetilde{\to} \mathcal{H}_2 \ - \ \text{Intertwiner},$$

such that for every $h \in H$

$$\begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{\pi_1(h)} & \mathcal{H}_1 \\ \downarrow \alpha & & \downarrow \alpha \\ \mathcal{H}_2 & \xrightarrow{\pi_2(h)} & \mathcal{H}_2 \end{array}$$

i.e., $\alpha \circ \pi_1(h) = \pi_2(h) \circ \alpha$.

(II) Representation Theory

Definitions

We say that $(\pi_1, H, \mathcal{H}_1)$, $(\pi_2, H, \mathcal{H}_2)$ are equivalent, $\pi_1 \simeq \pi_2$, if

$$\exists\ \alpha : \mathcal{H}_1 \widetilde{\to} \mathcal{H}_2 \ - \ \text{Intertwiner},$$

such that for every $h \in H$

$$\begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{\pi_1(h)} & \mathcal{H}_1 \\ \downarrow \alpha & & \downarrow \alpha \\ \mathcal{H}_2 & \xrightarrow{\pi_2(h)} & \mathcal{H}_2 \end{array}$$

i.e., $\alpha \circ \pi_1(h) = \pi_2(h) \circ \alpha$.

- Example: *DFT* is an intertwiner!

Rep'n Theory, cont.

- $H = V \times Z = \underbrace{\mathbb{Z}_N}_{T} \times \underbrace{\mathbb{Z}_N}_{W} \times \underbrace{\mathbb{Z}_N}_{Z}$

Rep'n Theory, cont.

- $H = V \times Z = \underbrace{\mathbb{Z}_N}_{T} \times \underbrace{\mathbb{Z}_N}_{W} \times \underbrace{\mathbb{Z}_N}_{Z}$
- Time representation: $\mathcal{H}_T = \mathbb{C}(T)$

$$\begin{cases} \pi_T : H \longrightarrow GL(\mathcal{H}_T); \\ \pi_T(\tau, \omega, z) = e^{\frac{2\pi i}{N}\{\frac{1}{2}\omega\tau + z\}} \cdot M_\omega \circ L_\tau. \end{cases}$$

Rep'n Theory, cont.

- $H = V \times Z = \underbrace{\mathbb{Z}_N}_{T} \times \underbrace{\mathbb{Z}_N}_{W} \times \underbrace{\mathbb{Z}_N}_{Z}$
- Time representation: $\mathcal{H}_T = \mathbb{C}(T)$

$$\begin{cases} \pi_T : H \to GL(\mathcal{H}_T); \\ \pi_T(\tau, \omega, z) = e^{\frac{2\pi i}{N}\{\frac{1}{2}\omega\tau + z\}} \cdot M_\omega \circ L_\tau. \end{cases}$$

- Frequency representation: $\mathcal{H}_W = \mathbb{C}(W)$

$$\begin{cases} \pi_W : H \to GL(\mathcal{H}_W); \\ \pi_W(\tau, \omega, z) = e^{\frac{2\pi i}{N}\{-\frac{1}{2}\omega\tau + z\}} \cdot M_\tau \circ L_{-\omega}. \end{cases}$$

Rep'n Theory, cont.

- $H = V \times Z = \underbrace{\mathbb{Z}_N}_{T} \times \underbrace{\mathbb{Z}_N}_{W} \times \underbrace{\mathbb{Z}_N}_{Z}$
- Time representation: $\mathcal{H}_T = \mathbb{C}(T)$

$$\begin{cases} \pi_T : H \to GL(\mathcal{H}_T); \\ \pi_T(\tau, \omega, z) = e^{\frac{2\pi i}{N}\{\frac{1}{2}\omega\tau + z\}} \cdot M_\omega \circ L_\tau. \end{cases}$$

- Frequency representation: $\mathcal{H}_W = \mathbb{C}(W)$

$$\begin{cases} \pi_W : H \to GL(\mathcal{H}_W); \\ \pi_W(\tau, \omega, z) = e^{\frac{2\pi i}{N}\{-\frac{1}{2}\omega\tau + z\}} \cdot M_\tau \circ L_{-\omega}. \end{cases}$$

- Fact:

$$DFT \circ \pi_T(h) = \pi_W(h) \circ DFT,$$

where

$$DFT[f](w) = \frac{1}{\sqrt{N}} \sum_{t \in \mathbb{Z}_N} f(t) e^{\frac{2\pi i}{N} wt}.$$

(III) FFT Algorithm

- Idea:

$$\begin{array}{ccc} \mathcal{H}_T & \xrightarrow{fast} & \mathcal{H}^\Lambda \\ {\scriptstyle slow}\downarrow{\scriptstyle DFT} & & \| \\ \mathcal{H}_W & \xleftarrow[fast]{} & \mathcal{H}^\Lambda \end{array}$$

(III) FFT Algorithm

- Idea:

$$\begin{array}{ccc} \mathcal{H}_T & \xrightarrow{fast} & \mathcal{H}^\Lambda \\ {\scriptstyle slow}\downarrow{\scriptstyle DFT} & & \| \\ \mathcal{H}_W & \xleftarrow[fast]{} & \mathcal{H}^\Lambda \end{array}$$

- More representation theory:

(III) FFT Algorithm

- Idea:

$$\begin{array}{ccc} \mathcal{H}_T & \xrightarrow{fast} & \mathcal{H}^\Lambda \\ {\scriptstyle slow}\downarrow{\scriptstyle DFT} & & \| \\ \mathcal{H}_W & \xleftarrow[fast]{} & \mathcal{H}^\Lambda \end{array}$$

- More representation theory:

Definition

A rep'n $(\pi, H, \mathcal{H})$ is <u>irreducible</u> if

$$\nexists \ 0 \neq \mathcal{H}' \subsetneq \mathcal{H}$$

such that

$$\pi(h) \cdot \mathcal{H}' \subset \mathcal{H}', \quad \forall h \in H.$$

FFT Algorithm: more RT

Theorem (Stone–von Neumann)

If $(\pi_j, H, \mathcal{H}_j)$, $j = 1, 2$, *are irreducible representations of the Heisenberg group* $H = V \times Z$, *with*

$$\pi_j(z) = e^{\frac{2\pi i}{N} z} \cdot Id_{\mathcal{H}_j}, \ \forall z \in Z,$$

then $\pi_1 \simeq \pi_2$ *are equivalent, i.e.,* $\exists\ \alpha : \mathcal{H}_1 \widetilde{\to} \mathcal{H}_2$ *such that*

$$\alpha \circ \pi_1(h) = \pi_2(h) \circ \alpha, \quad \forall h \in H. \tag{1}$$

FFT Algorithm: more RT

Theorem (Stone–von Neumann)

If $(\pi_j, H, \mathcal{H}_j)$, $j = 1, 2$, *are irreducible representations of the Heisenberg group* $H = V \times Z$, *with*

$$\pi_j(z) = e^{\frac{2\pi i}{N} z} \cdot Id_{\mathcal{H}_j}, \ \forall z \in Z,$$

then $\pi_1 \simeq \pi_2$ *are equivalent, i.e.,* $\exists\ \alpha : \mathcal{H}_1 \widetilde{\to} \mathcal{H}_2$ *such that*

$$\alpha \circ \pi_1(h) = \pi_2(h) \circ \alpha, \quad \forall h \in H. \tag{1}$$

Theorem (Schur's lemma)

If $\pi_1 \simeq \pi_2$ *equivalent irreducible representations, and if* α, α' *satisfy equation (1), then*

$$\alpha = c \cdot \alpha', \quad \text{for some scalar } c.$$

FFT Algorithm: Models of Heisenberg Rep'n

Examples

(1) Time model: $V \supset T = \{(t, 0);\ t \in \mathbb{Z}_N\}$

$$\pi_T : H \to GL(\mathcal{H}_T).$$

(2) Frequency model: $V \supset W = \{(0, w);\ w \in \mathbb{Z}_N\}$

$$\pi_W : H \to GL(\mathcal{H}_W).$$

Corollary

$DFT : \mathcal{H}_T \to \mathcal{H}_W$ — unique (up to a scalar) intertwiner between π_T and π_W.

FFT Algorithm: Models of Heisenberg Rep'n

Examples

(3) W–invariant model:
Space:

$$\mathcal{H}^W = \begin{cases} f : H \to \mathbb{C}, \\ f(w \cdot h) = f(h), \quad \forall\ w \in W,\ h \in H, \\ f(z \cdot h) = e^{\frac{2\pi i}{N} z} \cdot f(h), \quad \forall\ z \in Z,\ h \in H. \end{cases}$$

Action:

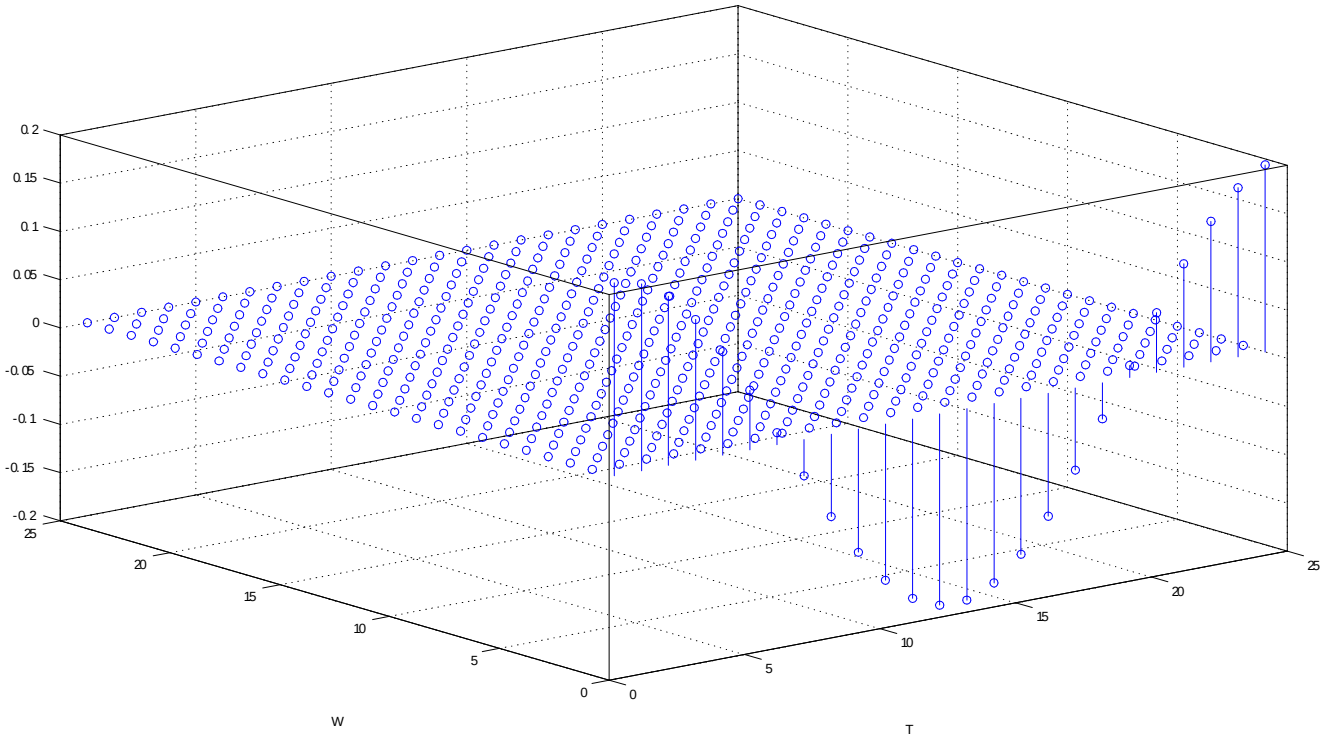
$$\begin{cases} \pi^W : H \to GL(\mathcal{H}^W), \\ [\pi^W(h')f](h) = f(h \cdot h'). \end{cases}$$

- Remark: We have a natural identification $\mathcal{H}_T = \mathcal{H}^W$, given by

$$f \mapsto \widetilde{f}(wtz) = e^{\frac{2\pi i}{N} z} \cdot f(t).$$

Think on W-invariant functions as functions on T

A function $f(t)$ on $T = \{(t, 0, 0);\ t \in \mathbb{Z}/5^2\}$



FFT Algorithm: Models of Heisenberg Rep'n

Examples

(4) T–invariant model:

Space:

$$\mathcal{H}^T = \begin{cases} g : H \to \mathbb{C}, \\ g(t \cdot h) = g(h), \quad \forall\ t \in T,\ h \in H, \\ g(z \cdot h) = e^{\frac{2\pi i}{N} z} \cdot g(h), \quad \forall\ z \in Z,\ h \in H. \end{cases}$$

Action:

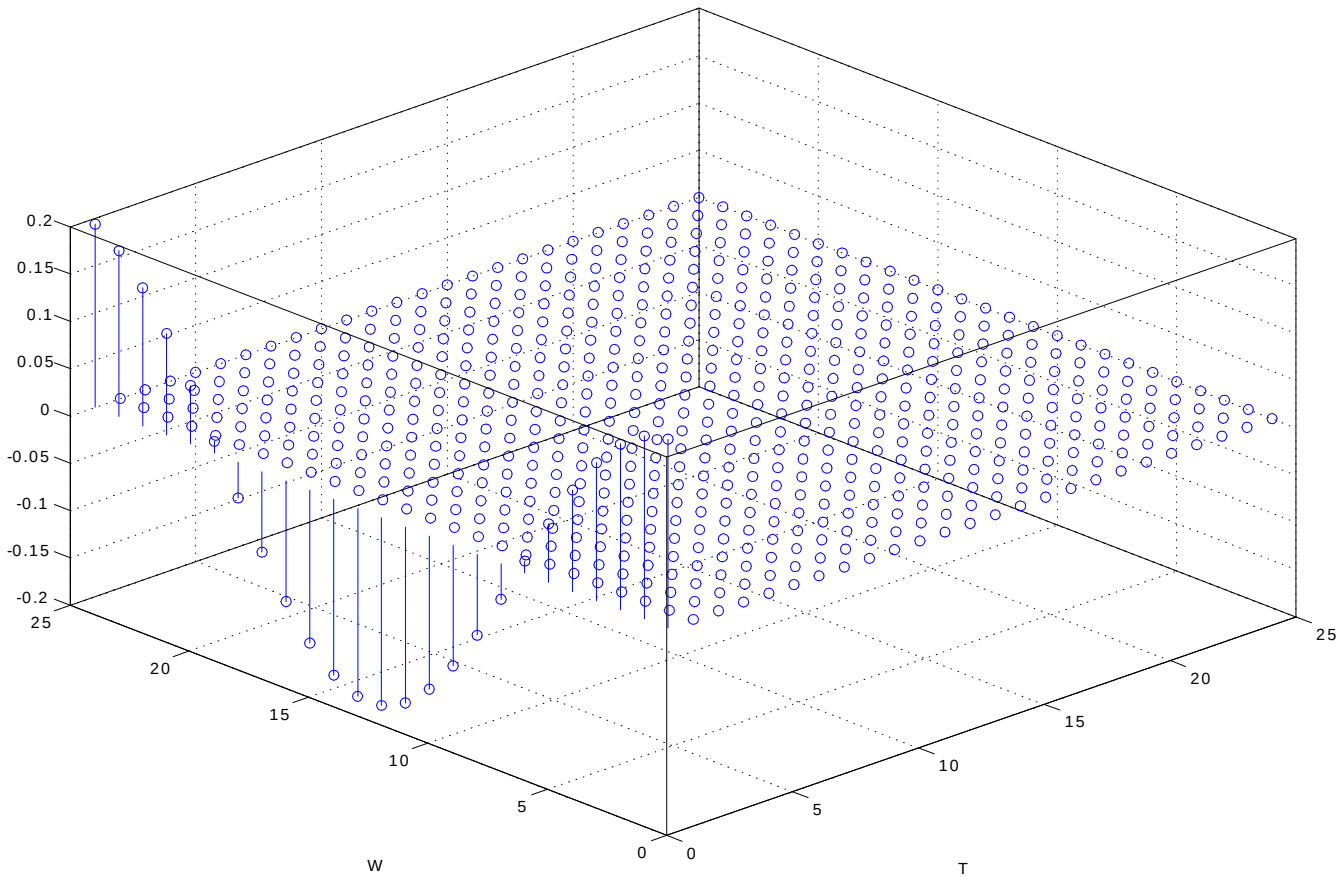
$$\begin{cases} \pi^T : H \to GL(\mathcal{H}^T), \\ [\pi^T(h')g](h) = g(h \cdot h'). \end{cases}$$

- Remark: We have a natural identification $\mathcal{H}_W = \mathcal{H}^T$, given by

$$g \mapsto \widetilde{g}(twz) = e^{\frac{2\pi i}{N} z} \cdot g(w).$$

Think on T-invariant functions as functions on W

A function $g(w)$ on $W = \{(0, w, 0);\ \ w \in \mathbb{Z}/5^2\}$



FFT Algorithm: The Arithmetic Model

Examples

(5) Arithmetic model: $N = p^2$

Lagrangian: $V = \mathbb{Z}/p^2 \times \mathbb{Z}/p^2 \supset \Lambda = \{(p \cdot a,\ p \cdot b)\}$

Space:

$$\mathcal{H}^{\Lambda} = \left\{ \begin{array}{c} F : H \to \mathbb{C}, \\ F(\lambda \cdot h) = F(h), \quad \forall\ \lambda \in \Lambda,\ h \in H, \\ F(z \cdot h) = e^{\frac{2\pi i}{N} z} \cdot F(h), \quad \forall\ z \in Z,\ h \in H. \end{array} \right.$$

Action:

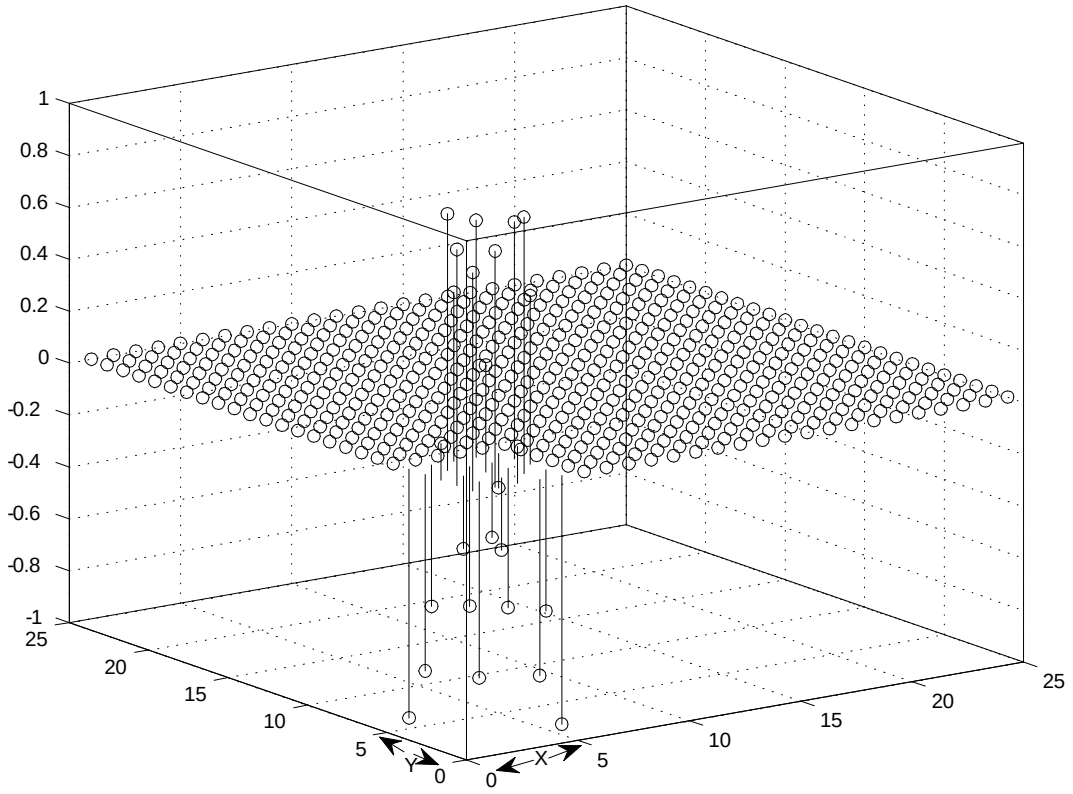
$$\pi^{\Lambda} : H \to GL(\mathcal{H}^{\Lambda}), \qquad [\pi^{\Lambda}(h')F](h) = F(h \cdot h').$$

- Remark: We have a natural identification of $\mathcal{H}^{\Lambda}$ with functions $F(x, y)$ on $\{0, ..., p-1\} \times \{0, ..., p-1\}$

$$F \mapsto \widetilde{F}(\lambda \cdot (x, y, 0) \cdot z) = e^{\frac{2\pi i}{N} z} \cdot F(x, y),\ \lambda \in \Lambda.$$

Think on Lambda-invariant functions as functions on {0,...,p-1} x {0,...,p-1}

A function $F(x, y)$ on $(x, y) \in \{0, ..., 4\} \times \{0, ..., 4\}$, $p = 5$,



FFT: The Algorithm

- Algorithm: Case $N = p^2$

$$
\begin{array}{ccc}
\mathcal{H}_T & \xrightarrow[p^4]{DFT} & \mathcal{H}_W \\
\Big\downarrow{\scriptstyle saw} & & \Big\uparrow{\scriptstyle saw} \\
\mathcal{H}^W & & \mathcal{H}^T \\
\Big\downarrow{\scriptstyle \sum_{\lambda\in\Lambda} f(\lambda h)} \; ? & & ? \; \Big\uparrow{\scriptstyle \sum_{t\in T} F(tw)} \\
\mathcal{H}^\Lambda & = & \mathcal{H}^\Lambda
\end{array}
$$

FFT: The Algorithm

- Algorithm: Case $N = p^2$

$$\begin{array}{ccc}
\mathcal{H}_T & \xrightarrow[p^4]{DFT} & \mathcal{H}_W \\
\Big\downarrow{\scriptstyle saw} & & \Big\uparrow{\scriptstyle saw} \\
\mathcal{H}^W & & \mathcal{H}^T \\
\Big\downarrow{\scriptstyle \sum_{\lambda\in\Lambda} f(\lambda h)} \Big| ? & & ? \Big| \Big\uparrow{\scriptstyle \sum_{t\in T} F(tw)} \\
\mathcal{H}^\Lambda & = & \mathcal{H}^\Lambda
\end{array}$$

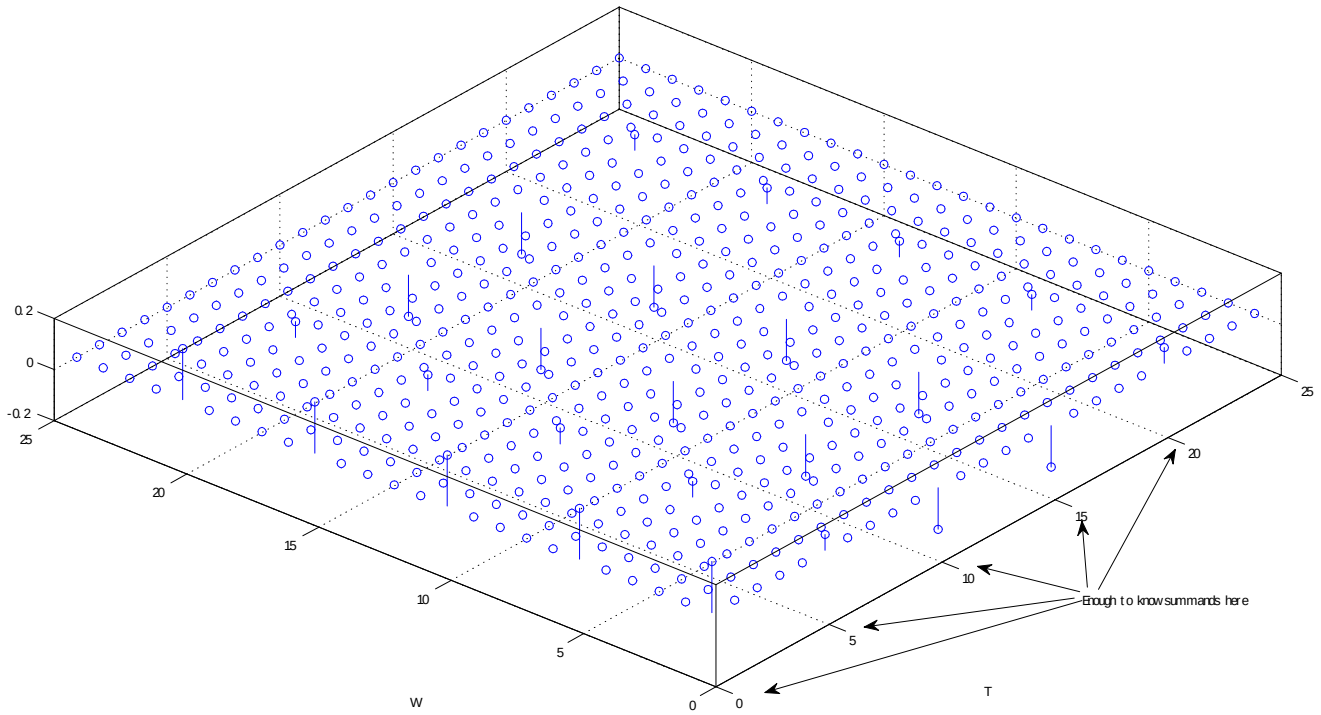
- Observation **(the saving!)**: For $f \in \mathcal{H}^W$, if $\lambda, \lambda' \in \Lambda$ differ by an element of W, i.e., $\lambda = w\lambda'$, then

$$f(\lambda h) = f(w\lambda' h) = f(\lambda' h), \quad \forall h \in H.$$

FFT Algorithms: The Summands

So for a fixed h it is enough to know the summands $f(\lambda h)$, in the transform, only for $\lambda \in \Lambda / [\Lambda \cap W] = \{(0,0),\ (p,0), ..., ((p-1)p, 0)\}$.

- Example for $p = 5$



FFT Algorithm: The Saving

- So we have

$$\sum_{\lambda\in\Lambda} f(\lambda h) = \sum_{\lambda\in\Lambda/[\Lambda\cap W]} f(\lambda h)\cdot \#(\Lambda\cap W).$$

Only $p = p^2/p = \#(\Lambda/[\Lambda\cap W])$ summands !!!.

FFT Algorithm: The Saving

- So we have

$$\sum_{\lambda\in\Lambda} f(\lambda h) = \sum_{\lambda\in\Lambda/[\Lambda\cap W]} f(\lambda h)\cdot\#(\Lambda\cap W).$$

  Only $p = p^2/p = \#(\Lambda/[\Lambda\cap W])$ summands !!!.

- Complexity of the algorithm

$$\underbrace{p^2}_{\text{values of h}} \cdot \underbrace{p}_{\text{summands}} + \underbrace{p^2\cdot p}_{\text{second operator}} = p\cdot p^2\cdot(1+1) = \underbrace{p}_{\text{constant}}\cdot N\cdot\log(N).$$

FFT Algorithm: Schur's lemma

Denote the intertwiners by

$$FFT^{\Lambda,W} : \mathcal{H}^W \to \mathcal{H}^\Lambda, \text{ and } FFT^{T,\Lambda} : \mathcal{H}^\Lambda \to \mathcal{H}^T.$$

- Why

$$DFT = FFT^{T,\Lambda} \circ FFT^{\Lambda,W} \ ? \tag{2}$$

FFT Algorithm: Schur's lemma

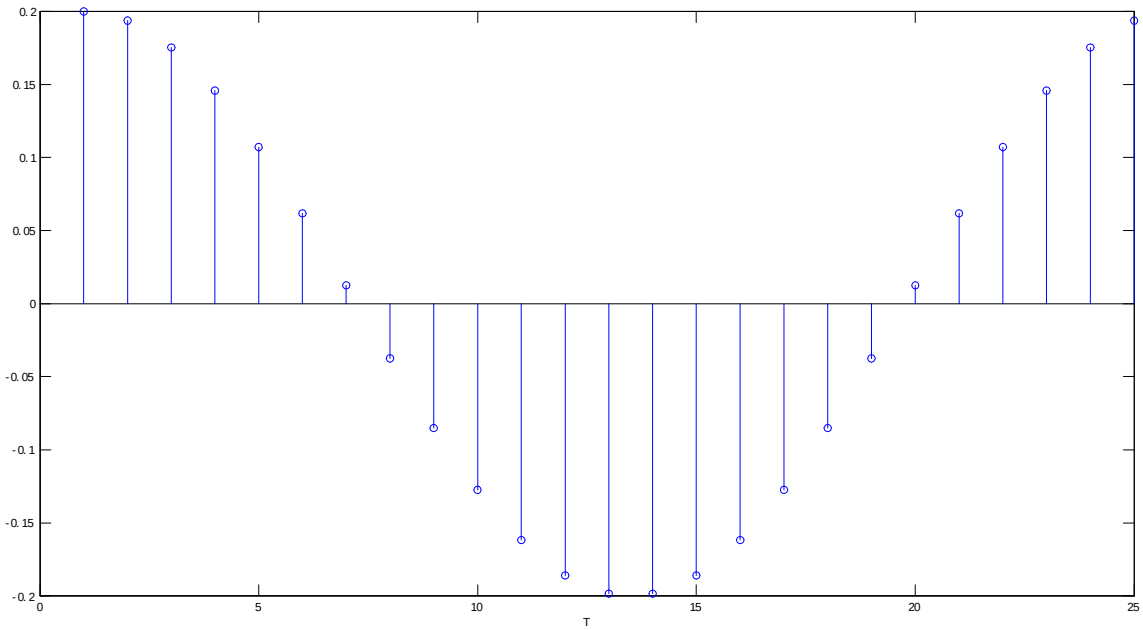
Denote the intertwiners by

$$FFT^{\Lambda,W} : \mathcal{H}^W \to \mathcal{H}^\Lambda, \text{ and } FFT^{T,\Lambda} : \mathcal{H}^\Lambda \to \mathcal{H}^T.$$

- Why
$$DFT = FFT^{T,\Lambda} \circ FFT^{\Lambda,W} \ ? \tag{2}$$
- In fact by Schur's lemma: Both sides of (2) intertwine π^W and π^T, so
$$DFT = c \cdot FFT^{T,\Lambda} \circ FFT^{\Lambda,W}$$
for some scalar c. Easy to compute $c = \frac{1}{p}$.

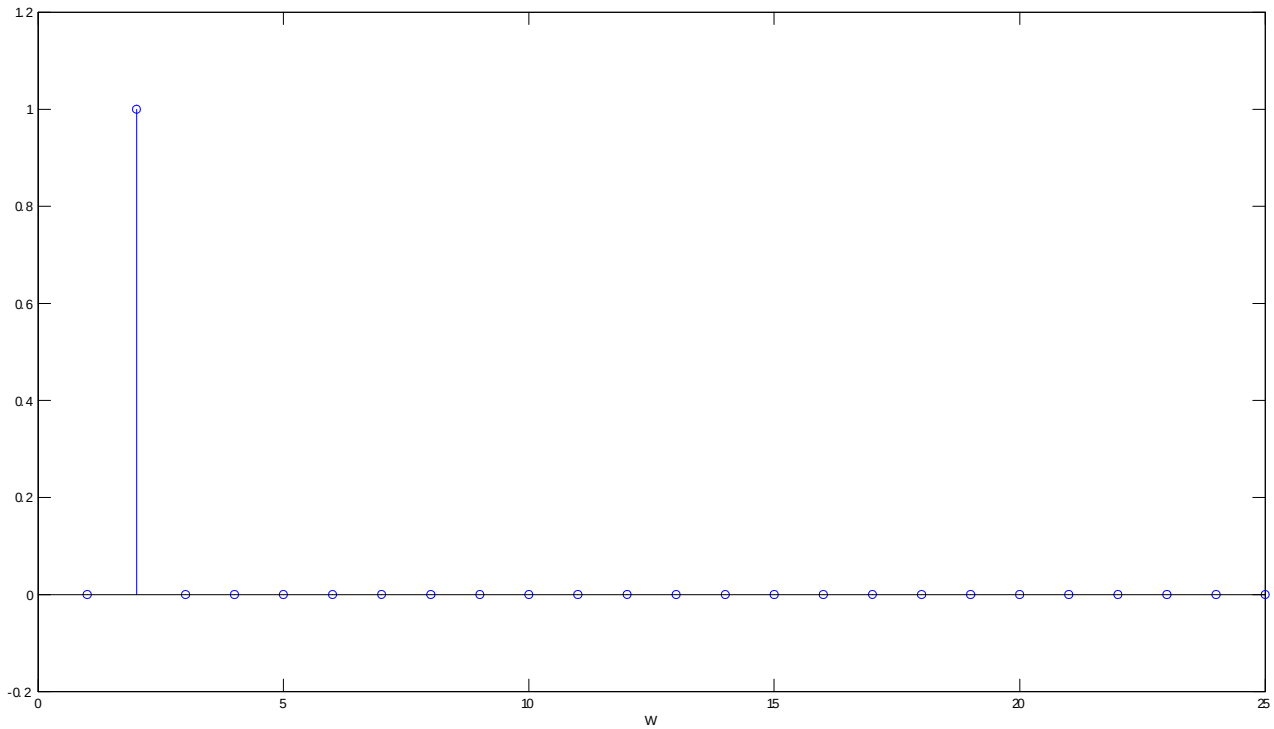
Numerics for n=25

Start with the exponent function $f(t) = e^{\frac{2\pi i}{25}t}/5$ on $T = \mathbb{Z}/25$



Numerics for n=25 : FFT(f)

Apply FFT and get $g(w) = FFT[f](w)$ on $W = \mathbb{Z}/25$



Indeed this is $g(w) = \delta_1(w)$.

FFT Algorithm: General

- $N = p^k$, $k \geq 1$

FFT Algorithm: General

- $N = p^k$, $k \geq 1$
- $V = \mathbb{Z}/p^k \times \mathbb{Z}/p^k$

FFT Algorithm: General

- $N = p^k$, $k \geq 1$
- $V = \mathbb{Z}/p^k \times \mathbb{Z}/p^k$
- Good Lagrangian sequence ("interpolating" W and T)

$$\Lambda_\bullet : \qquad \Lambda_j = \{(p^{k-j} \cdot a, p^j \cdot b)\}, \qquad j = 0, ..., k,$$

with LARGE intersections

$$\# \ \Lambda_j \cap \Lambda_{j+1} = p^{k-1}.$$

FFT Algorithm: General

- $N = p^k$, $k \geq 1$
- $V = \mathbb{Z}/p^k \times \mathbb{Z}/p^k$
- Good Lagrangian sequence ("interpolating" W and T)

$$\Lambda_\bullet : \qquad \Lambda_j = \{(p^{k-j} \cdot a, p^j \cdot b)\}, \qquad j = 0, ..., k,$$

with LARGE intersections

$$\# \ \Lambda_j \cap \Lambda_{j+1} = p^{k-1}.$$

- FFT Algorithm

$$\begin{aligned} \mathcal{H}^W \ \overset{FFT^{\Lambda_1,W}}{\longrightarrow} \ \mathcal{H}^{\Lambda_1} \quad &\longrightarrow \quad \cdots \longrightarrow \mathcal{H}^{\Lambda_{k-1}} \ \overset{FFT^{T,\Lambda_k}}{\longrightarrow} \ \mathcal{H}^T \\ DFT \quad &= \quad \frac{1}{\sqrt{p^k}} \cdot FFT^{T,\Lambda_{k-1}} \circ \cdots \circ FFT^{\Lambda_1,W}, \end{aligned}$$

with complexity

$$\underbrace{p^k}_{\text{values of } h} \cdot \{ \ \underbrace{k}_{k \text{ operators}} \ \cdot \underbrace{\# \ \Lambda_{j+1}/(\Lambda_j \cap \Lambda_{j+1})}_{p \text{ summands}} \} = \underbrace{p}_{\text{constant}} \ \underbrace{\cdot p^k}_{N} \cdot \underbrace{\log(p^k)}_{\log(N)}.$$

Conclusion

- Conclusion:

$$\Lambda_\bullet \Longrightarrow \text{Cooley–Tukey } \textit{FFT}$$

THANK YOU

Thank You!