Random Walks on Hyperbolic Groups III

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January 2014

I. Background: Hyperbolic Geometry

Hyperbolic Metric Space: A geodesic metric space such that for some $\delta > 0$ all geodesic triangles are $\delta-$thin.

Examples:

(A) The hyperbolic plane.

(B) The $d-$regular tree.

Definition: A finitely generated group Γ is hyperbolic (also called word-hyperbolic) if its Cayley graph (when given the usual graph metric) is a hyperbolic metric space.

Lemma: If Γ is hyperbolic with respect to generators A then it is hyperbolic with respect to any finite symmetric generating set.

Geometry of the Hyperbolic Plane

Halfplane Model: $\mathbb{H} = \{x + iy \,:\, y > 0\}$ with metric ds/y
Disk Model: $\mathbb{D} = \{re^{i\theta} \,:\, r < 1\}$ with metric $4\,ds/(1 - r^2)$.

Miscellaneous Facts:

(0) Disk and halfplane models are isometric.

(1) Isometries are linear fractional transformations that fix $\partial\mathbb{H}$ or $\partial\mathbb{D}$.

(2) Isometry group is transitive on $\mathbb{H}$ and $\partial\mathbb{H} \times \partial\mathbb{H}$.

(3) Geodesics are circular arcs that intersect $\partial\mathbb{H}$ or $\partial\mathbb{D}$ orthogonally.

(4) Circle of radius t has circumference $\asymp e^t$.

Fuchsian Groups

Fuchsian group: A discrete group of isometries of the hyperbolic plane $\mathbb{H}$. Examples: Surface group (fundamental group of closed surface of genus ≥ 2), free group $\mathbb{F}_d$, triangle groups.

Note: Isometries of $\mathbb{H}$ are linear fractional transformations $z \mapsto \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{R})$. Hence, a Fuchsian group is really a discrete subgroup of $SL(2, \mathbb{R})$

Fact: The Cayley graph of a finitely generated Fuchsian group can be quasi-isometrically embedded in $\mathbb{H}$. This implies that any Fuchsian group is hyperbolic.

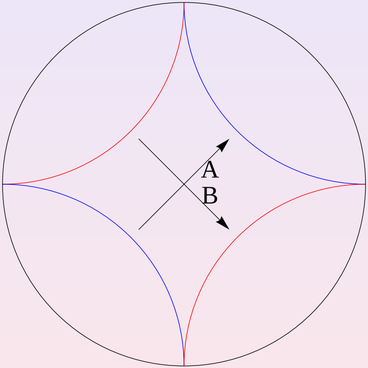
Consequence: Geodesics of the Cayley graph (word metric) track hyperbolic geodesics at bounded distance.

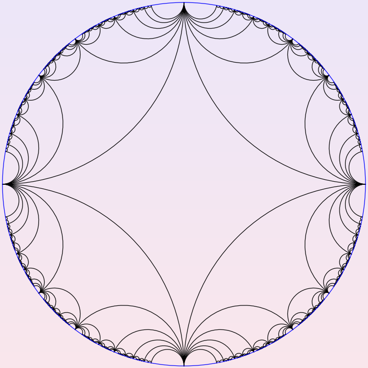
Fuchsian Groups

Hyperbolic Isometry: Any isometry ψ of $\mathbb{D}$ that is conjugate by map $\Phi : \mathbb{H} \to \mathbb{D}$ to a linear fractional transformation $z \mapsto \lambda z$ of $\mathbb{H}$. The geodesic from $\Phi(0)$ to $\Phi(\infty)$ is the axis of ψ, and $\Phi(0), \Phi(\infty)$ are the fixed points.

Proposition: If Γ is a co-compact Fuchsian group then Γ contains hyperbolic elements, and fixed-point pairs ξ, ζ of such elements are dense in $\partial\mathbb{D} \times \partial\mathbb{D}$.

Free Groups are Fuchsian





Other Examples of Hyperbolic Groups

$\mathbb{Z}$ is hyperbolic.

Free products $G_1 * G_2 * \cdots * G_d$ of finite groups are hyperbolic.

Fundamental groups of compact Riemannian manifolds with strictly negative sectional curvature are hyperbolic.

Discrete groups of isometries of $n-$dimensional hyperbolic space $\mathbb{H}^n$.

Theorem (Bonk-Schramm): If Γ is hyperbolic then for any finite symmetric generating set A the Cayley graph $G(\Gamma; A)$ is quasi-isometric to a convex subset of $\mathbb{H}^n$.

Geometric Boundary and Gromov Compactification

Two geodesic rays $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ are equivalent if there exists $k \in \mathbb{Z}$ such that for all large n,

$$d(x_n, y_{n+k}) \leq 2\delta.$$

Geometric Boundary $\partial\Gamma$: Set of equivalence classes of geodesic rays.

Topology on $\Gamma \cup \partial\Gamma$: Basic open sets:

(a) singletons $\{x\}$ with $x \in \Gamma$; and

(b) sets $B_m(\xi)$ with $m \geq 1$ and $\xi \in \partial\Gamma$ where $B_m(\xi) =$ set of $x \in \Gamma$ and $\zeta \in \partial\Gamma$ such that there exists geodesic rays (x_n) and (y_n) with initial points $x_0 = y_0 = 1$, endpoints ξ and ζ (or ξ and x), and such that

$$d(x_j, y_j) \leq 2\delta \quad \forall\, j \leq m$$

Non-elementary Hyperbolic Group: $|\partial\Gamma| = \infty$.

Geometric Boundary and Gromov Compactification

Basic Facts

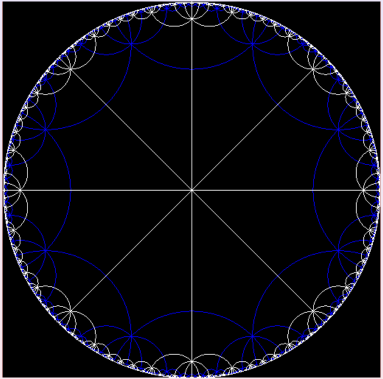
Proposition 1: $\partial\Gamma$ is compact in the Gromov topology.

Proposition 2: Γ acts by homeomorphisms on $\partial\Gamma$, and if Γ is nonelementary then every $\Gamma-$orbit is dense in $\partial\Gamma$.

Proposition 3: If Γ is a finitely generated, nonelementary hyperbolic group then Γ is nonamenable.

In fact, the action of Γ on $\partial\Gamma$ has no invariant probability measure.

Geometric Boundary and Gromov Compactification



Fact: If Γ is a co-compact Fuchsian group (i.e., if $\mathbb{H}/\Gamma$ is compact) then the geometric boundary is homeomorphic to the circle.

Fact: If Γ is a co-compact Fuchsian group then the set of pairs $(\xi-, \xi_+)$ of fixed points of hyperbolic elements of Γ is dense in $\partial\mathbb{D} \times \partial\mathbb{D}$.

Convergence to the Boundary

Theorem: Let X_n be a symmetric, irreducible FRRW on a nonamenable hyperbolic group Γ. Then with probability one the sequence X_n converges to a (random) point $X_\infty \in \partial\Gamma$.

Proof: (Sketch) Since Γ is nonamenable the random walk has positive speed. Since the random walk has bounded step size, the (word) distance between successive points X_n and X_{n+1} is $O(1)$. Now use:

Lemma: If x_n is any sequence of points such that $d(1, x_n)/n \to \alpha > 0$ and $d(x_n, x_{n+1})$ is bounded then x_n converges to a point of the Gromov boundary.

Convergence to the Boundary

Theorem: Let X_n be a symmetric, irreducible FRRW on a nonamenable hyperbolic group Γ. Then with probability one the sequence X_n converges to a (random) point $X_\infty \in \partial\Gamma$.

Proposition: The distribution of X_∞ is nonatomic, and attaches positive probability to every nonempty open set $U \subset \partial\Gamma$.

Note: The result is due to Furstenberg (?). For an exposition see Kaimanovich, *Ann. Math.* v. 152

Visual Metric on $\partial\Gamma$

Visual Metric: A metric d_a on $\partial\Gamma$ such that for any $\xi, \zeta \in \partial\Gamma$, any bi-infinite geodesic γ from ξ to ζ, and any vertex y on γ minimizing distance to 1,

$$C_1 a^{-d(1,y)} \leq d_a(\xi, \zeta) \leq C_2 a^{-d(1,y)}$$

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$$C_1 a^{-d(1,y)} \leq d_a(\xi, \zeta) \leq C_2 a^{-d(1,y)}$$

Proposition: For some $a > 1$ a visual metric exists.

Remark: For the hyperbolic plane $\mathbb{D}$, the Euclidean metric on $\partial\mathbb{D}$ is a visual metric.

Ledrappier-Kaimanovich Formula

Billingsley Dimension: Let ν be a probability measure on metric space $(\mathcal{Y}, d)$. Define

$$\dim(\nu) = \inf\{\text{H-dim}(A) \,:\, \nu(A) = 1\}.$$

Theorem: (Le Prince; BHM) Let Γ be a hyperbolic group with geometric boundary $\partial\Gamma$ and visual metric d_a on $\partial\Gamma$. For any FRRW on Γ with Avez entropy h, speed ℓ, and exit measure ν_1,

$$\dim(\nu_1) = \frac{1}{\log a} \frac{h}{\ell}$$

Theorem: (Furstenberg) For any co-compact Fuchsian group Γ there is a symmetric probability measure μ on Γ such that the RW with step distribution μ has exit distribution absolutely continuous relative to Lebesgue on S^1. The measure μ does not have finite support.

Conjecture: For any finite symmetric generating set A there is a constant $C_A < \dim_H(\partial\Gamma)$ such that for any symmetric FRRW with step distribution supported by A

$$\dim(\nu_1) \leq C_A.$$

II. Martin Kernel and Martin Boundary

Martin Kernel:

$$k_y(x) = K_r(x, y) = \frac{G_r(x, y)}{G_r(1, y)} \quad \text{where}$$

$$G_r(x, y) = \sum_{n=0} r^n P^x\{X_n = y\}$$

Martin Compactification $\hat{\Gamma}$: Unique minimal compactification of Γ to which each function $y \mapsto k_y(x)$ extends continuously.

Martin Boundary: Set $\partial\hat{\Gamma}$ of all pointwise limits $\lim_{n\to\infty} k_{y_n}(\cdot)$ not already included in $\{k_y\}_{y\in\Gamma}$. The functions in $\partial\hat{\Gamma}$ are $r-$harmonic.

Martin Kernel and Martin Boundary

Theorem: (Series-Ancona-Gouezel-Lalley) Let Γ be a nonelementary hyperbolic group. Then for any symmetric FRRW on Γ and any $1 \leq r \leq R$ the Martin boundary is homeomorphic to the geometric boundary.

Series: $r = 1$, Fuchsian groups
Ancona: $r < R$, Hyperbolic groups
Gouezel-Lalley: $r = R$, Fuchsian groups
Gouezel: $r = R$, Hyperbolic groups

Martin Kernel and Martin Boundary

Theorem: (GL) Let Γ be a nonelementary hyperbolic group. Then for any symmetric FRRW on Γ $\exists\, \beta < 1$ such that for every $1 \leq r \leq R$ and any geodesic ray $1 y_1 y_2 y_3 \cdots$ converging to a point $\xi \in \partial\Gamma$ of the geometric boundary,

$$\left| \frac{G_r(x, y_n)}{G_r(1, y_n)} - K_r(x, \zeta) \right| \leq C_x \beta^n.$$

Consequently, for each $x \in \Gamma$ the function $(r, \xi) \mapsto K_r(x, \xi)$ is Hölder continuous relative to visual metric on $\partial\Gamma$.

Martin Kernel and Martin Boundary

Question: Is the Martin boundary of a symmetric, FRRW on a co-compact lattice of a connected semisimple Lie group with finite center determined, up to homeomorphism type, by the ambient Lie group?

Question: Is the Martin boundary of a symmetric, FRRW on a nonamenable discrete group determined, up to homeomorphism type, by the group.

Ancona Inequalities

Key to the Martin Boundary:

Theorem A: (Ancona Inequalities) Let Γ be a nonelementary hyperbolic group. Then for any symmetric FRRW on Γ with spectral radius $\varrho = 1/R$ there exists $C < \infty$ such that for any $x, y, z \in \Gamma$, if y lies on the geodesic segment from x to z then for all $1 \leq r \leq R$,

$$G_r(x, z) \leq C G_r(x, y) G_r(y, z)$$

Note: Reverse inequality with $C = 1$ is trivial. The two inequalities imply that the multiplicative relation exploited in the Dynkin-Malyutov proof almost holds.

Exponential Decay of the Green's function

Theorem B: (Exponential Decay of Green's Function) Let Γ be a nonelementary hyperbolic group. Then for any symmetric FRRW on Γ there exist $C < \infty$ and $0 < \beta < 1$ such that for all $1 \leq r \leq R$ and all $x \in \Gamma$,

$$G_r(1, x) \leq C\beta^{d(1,x)}$$

Remark: For an irreducible random walk it is always the case that the Green's function decays no faster than exponentially in distance.

Explanation: Assume for simplicity that the step distribution gives probability $\geq \alpha > 0$ to each generator of Γ. Then for $d(x, y) = m$ there is a path of length m from x to y with probability $\geq \alpha^m$, so

$$G_r(x, y) \geq r^m \alpha^m.$$

Exponential Decay of the Green's Function

Objective: Prove Theorems A–B for nearest neighbor, symmetric random walk on a co-compact Fuchsian group Γ.

Assumption: Henceforth Γ is a co-compact Fuchsian group, and only symmetric, nearest neighbor random walks will be considered.

Preliminary Observations:

(1) $\lim_{d(1,x)\to\infty} G_R(1,x) = 0$.

(2) $G_R(1,xy) \geq G_R(1,x)G_R(1,y)$

Proof of (1): Backscattering argument: Concatenating any path from 1 to x with path from x to 1 gives path from 1 to 1 of length $\geq 2d(1,x)$. Hence,

$$\sum_{n=2d(1,x)}^{\infty} R^n p^n(1,1) \geq G_R(1,x)^2 / G_R(1,1).$$

Exponential Decay of the Green's Function

Key Notion: A barrier is a triple (V, W, B) consisting of non-overlapping halfplanes V, W and a set B disjoint from $V \cup W$ such that every path from V to W passes through B; and

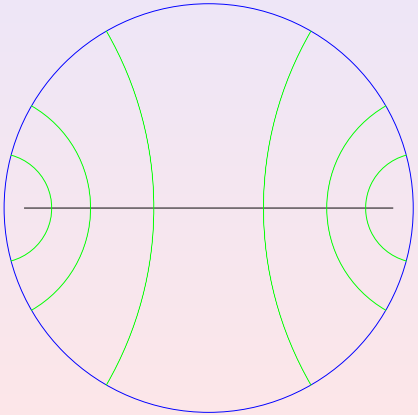
$$\max_{x \in V} \sum_{y \in B} G_R(x, y) \leq \frac{1}{2}.$$

Theorem C: For any two points $\xi \neq \zeta \in \partial\mathbb{D}$ there exists a barrier separating ξ and ζ.

Corollary: $\exists \varepsilon > 0$ such that any two points $x, y \in \Gamma$ are separated by $[\varepsilon d(x, y)]$ disjoint barriers.

Exponential Decay of the Green's Function

Barriers $\Longrightarrow$ exponential decay.



Explanation: Existence of barriers and compactness of $\partial\mathbb{D}$ implies that $\exists\,\varepsilon > 0$ such that for any $x \in \Gamma$ with $m = \mathrm{dist}(1, x)$ sufficiently large there are εm non-overlapping barriers B_i separating 1 from x. Hence,

$$G_r(1, x) \le -\sum_{z_i \in B_i} \prod_i G_r(z_i, z_{i+1})$$
$$\le 2^{-\varepsilon m}$$

Existence of Barriers

Strategy: Use random walk paths to build barriers.

Lemma: $E^1 G_R(1, X_n) \leq G_R(1, 1)^2 R^{-n}$

Proof: Paths from 1 to x can be concatenated with paths from x to 1 to yield paths from 1 to 1. Hence, by symmetry,
(i.e.,$F_R(1, x) = F_R(x, 1)$)

$$
\begin{aligned}
G_R(1,1) &\geq \sum_{k=n}^{\infty} R^k P^1\{X_k = 1\} \\
&\geq \sum_{x} R^n P^1\{X_n = x\} F_R(1, x) \\
&= R^n E^1 F_R(X_n, 1) \\
&= R^n E^1 G_R(1, X_n) / G_R(1, 1)
\end{aligned}
$$

where $F_R(1, x)$ is the first-passage generating function .

Existence of Barriers

Strategy: Use random walk paths to build barriers.

Lemma: $E^1 G_R(1, X_n) \leq G_R(1,1)^2 R^{-n}$

Corollary: If X_n and Y_n are independent versions of the random walk, both started at $X_0 = Y_0 = 1$, then

$$\begin{aligned} E^{1,1} G_R(Y_m, X_n) &= E^{1,1} G_R(1, Y_m^{-1} X_n) \\ &= E^1 G_R(z, X_{m+n}) \\ &\leq G_R(1,1)^2 R^{-m-n} \end{aligned}$$

Existence of Barriers

Construction: Attach independent random walk paths to the random points X_m and Y_m to obtain two-sided random paths $(U_n)_{n\in\mathbb{Z}}$ and $(V_n)_{n\in\mathbb{Z}}$ such that

$$\sum_{n,n'\in\mathbb{Z}} EG_R(U_n, V_{n'}) \leq 4G_R(1,1)^2 R^{-2m}$$

Recall: Each random walk path a.s. converges to a point of $\partial\mathbb{D}$, and the exit distribution is nonatomic.

Existence of Barriers

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Recall: Each random walk path a.s. converges to a point of $\partial\mathbb{D}$, and the exit distribution is nonatomic.

Consequence: There exist two-sided paths $\{u_n\}_{n\in\mathbb{Z}}$ and $\{v_n\}_{n\in\mathbb{Z}}$ converging to distinct endpoints $\xi_1, \xi_2, \xi_3, \xi_4 \in \partial\mathbb{D}$ such that

$$\sum_{n,n'\in\mathbb{Z}} G_R(u_n, v_{n'}) \leq 4G_R(1,1)^2 R^{-2m} < \frac{1}{2}.$$

The endpoint pairs ξ_1, ξ_2 and ξ_3, ξ_4 determine nonempty open disjoint arcs of $\partial\mathbb{D}$ that are separated by the paths $\{u_n\}_{n\in\mathbb{Z}}$ and $\{v_n\}_{n\in\mathbb{Z}}$.

Existence of Barriers

Conclusion: There exist two-sided paths $(u_n)_{n\in\mathbb{Z}}$ and $(v_n)_{n\in\mathbb{Z}}$ separating disjoint open arcs J and J' of $\partial\mathbb{D}$ such that

$$\sum_n \sum_m G_R(u_n, v_m) < \varepsilon.$$

Let U and V be halfplanes on opposite sides of the paths $(u_n)_{n\in\mathbb{Z}}$ and $(v_n)_{n\in\mathbb{Z}}$. Then the triple $(U, V, (v_m)_{m\in\mathbb{Z}})$ is a barrier.

To obtain barriers separating arbitrary points $\xi, \zeta \in \mathbb{D}$, apply isometries $g \in \Gamma$.

Proof of the Ancona Inequalities

Theorem A: (Ancona Inequalities) Let Γ be a co-compact Fuchsian group. Then for any symmetric nearest neighbor RW on Γ with spectral radius $\varrho = 1/R$ there exists $C < \infty$ such that for any $x, y, z \in \Gamma$, if y lies on the geodesic segment from 1 to z then for all $1 \leq r \leq R$,

$$G_r(x,z) \leq CG_r(x,y)G_r(y,z)$$

Note: A. Ancona proved that $G_r(x,z) \leq C_r G_r(x,y)G_r(y,z)$ for $r < R$ using a coercivity technique. See

A. Ancona, *Positive harmonic functions and hyperbolicity*, Springer LNM vol. 1344.

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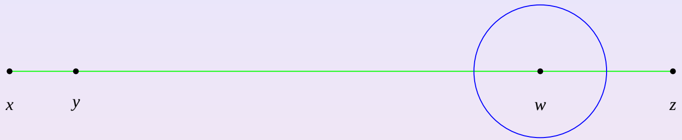
$$G_r(x, z) \leq C G_r(x, y) G_r(y, z)$$

Strategy: Let C_m be the max of $G_R(x, z)/G_R(x, y)G_R(y, z)$ over all triples x, y, z where y lies on the geodesic segment from 1 to z and $d(x, z) \leq m$. Since there are only finitely many possibilities, $C_m < \infty$.

To Show: $\sup C_m < \infty$
Will Show: $C_m / C_{(.9)m} \leq 1 + \varepsilon_m$ where $\sum \varepsilon_m < \infty$.

Proof of Ancona Inequalities



Place points x, y, w, z approximately along a geodesic at distances

$$d(x, y) = (.1)m$$
$$d(y, w) = (.7)m$$
$$d(w, z) = (.2)m$$

and let C be a circle of radius $\sqrt{m}$ centered at w. Assume m is large enough that $\sqrt{m} < (.1)m$.

Note: Any path from x to z must either enter C or go around C.

Proof of Ancona Inequalities

Fact: The hyperbolic circumference of C is $\approx e^{\sqrt{m/10}}$. Thus, a path from x to z that goes around C must pass through $\delta\sqrt{m}$ barriers.

Consequently, the contribution to the Green's function $G_R(x,z)$ from such paths is bounded above by

$$(1/2)^{\exp\{\delta\sqrt{m}\}}$$

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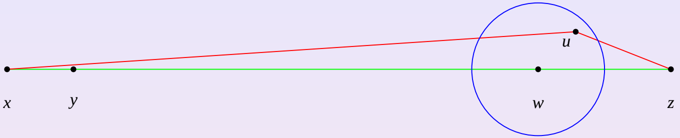
Proof of Ancona Inequalities



Any path from x to z that enters C must exit C a last time, at a point u inside C. Thus,

$$G_R(x,z) \leq 2^{-\exp\{\delta\sqrt{m}\}} + \sum_u G_R(x,u) G_R^*(u,z)$$

where $G_R^*(u,z)$ denotes sum over paths that do not re-enter C.

The distance from x to u is no larger than $(.9)m$, so

$$G_R(x,u) \leq C_{(.9)m} G_R(x,y) G_R(y,u)$$

Proof of Ancona Inequalities

Conclusion: Recall that there is a constant $\beta > 0$ such that $G_R(u, v) \geq \beta^m$ for any two points u, v at distance $\leq m$. Consequently,

$$\begin{aligned} G_R(x,z) &\leq 2^{-\exp\{\delta\sqrt{m}\}} + C_{(.9)m} G_R(x,y) \sum_u G_R(y,u) G_R^*(u,z) \\ &\leq 2^{-\exp\{\delta\sqrt{m}\}} + C_{(.9)m} G_R(x,y) G_R(y,z) \\ &\leq (1 + 2^{-\exp\{\delta\sqrt{m}\}}/\beta^m) C_{(.9)m} G_R(x,y) G_R(y,z) \end{aligned}$$

This proves

$$\boxed{C_m \leq C_{(.9)m}(1 + 2^{-\exp\{\delta\sqrt{m}\}}/\beta^m)}$$

Ancona $\Longrightarrow$ Convergence to Martin Kernel

Theorem: (GL) Let Γ be a nonelementary hyperbolic group. Then for any symmetric FRRW on Γ $\exists \beta < 1$ such that for every $1 \leq r \leq R$ and any geodesic ray $1 y_1 y_2 y_3 \cdots$ converging to a point $\xi \in \partial\Gamma$ of the geometric boundary,

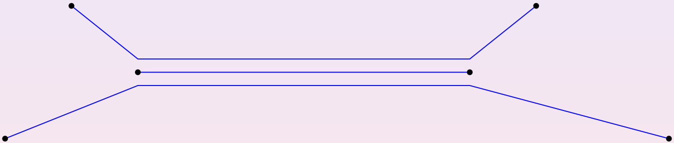
$$\left| \frac{G_r(x, y_n)}{G_r(1, y_n)} - K_r(x, \zeta) \right| \leq C_x \beta^n.$$

Consequently, for each $x \in \Gamma$ the function $(r, \xi) \mapsto K_r(x, \xi)$ is Hölder continuous relative to visual metric on $\partial\Gamma$.

Plan: Use Ancona inequalities to prove this following a template laid out by Anderson & Schoen and Ancona.

Convergence to the Martin Kernel

Shadowing: A geodesic segment $[x'y']$ shadows a geodesic segment $[xy]$ if every vertex on $[xy]$ lies within distance 2δ of $[x'y']$. If geodesic segments $[x'y']$ and $[x''y'']$ both shadow $[xy]$ then they are fellow-traveling along $[xy]$.



Proposition: $\exists\, 0 < \alpha < 1$ and $C < \infty$ such that if $[xy]$ and $[x'y']$ are fellow-traveling along a geodesic segment $[x_0 y_0]$ of length m then

$$\left| \frac{G_r(x,y)/G_r(x',y)}{G_r(x,y')/G_r(x',y')} - 1 \right| \leq C\alpha^m$$

Convergence to the Martin Kernel

Proposition: $\exists\, 0 < \alpha < 1$ and $C < \infty$ such that if $[xy]$ and $[x'y']$ are fellow-traveling along a geodesic segment $[x_0 y_0]$ of length m then

$$\left| \frac{G_r(x,y)/G_r(x',y)}{G_r(x,y')/G_r(x',y')} - 1 \right| \le C\alpha^m$$

Corollary: For any geodesic ray $1 y_1 y_2 y_3 \cdots$ converging to a point $\xi \in \partial\Gamma$ of the geometric boundary,

$$\left| \frac{G_r(x,y_n)}{G_r(1,y_n)} - K_r(x,\zeta) \right| \le C_x \alpha^n.$$

Consequently, for each $x \in \Gamma$ the function $(r,\xi) \mapsto K_r(x,\xi)$ is Hölder continuous relative to visual metric on $\partial\Gamma$.

Preliminary: Poisson Integral Formula

Restricted Green's Function: Let Ω be a subset of the Cayley graph, let $x \in \Omega$ and $y \notin \Omega$. Define the restricted Green's function to be the sum over all paths γ from $x \to y$ that remain in Ω until last step:

$$G_r(x, y; \Omega) = \sum_{\text{paths } x \to y \text{ in } \Omega} r^{|\gamma|} p(|\gamma|)$$

Poisson Integral Formula: Let Ω be a finite set and $u : \Gamma \to \mathbb{R}_+$ be a nonnegative function that is $r-$harmonic in Ω. Then for any $r \leq R$

$$u(x) = \sum_{y \notin \Omega} G_r(x, y; \Omega) u(y) \quad \forall\, x \in \Omega$$

Consequently, if u is bounded in Ω and $r \leq R$ then the formula holds also for infinite Ω.

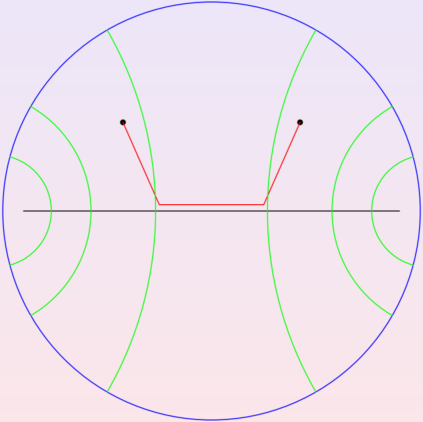
Ancona Inequalities for Restricted Green's Function

Proposition: Assume that Γ is a co-compact Fuchsian group and that its Cayley graph is embedded in $\mathbb{D}$. Let Ω be any halfplane, and for any $x, y, z \in \Omega$ such that y lies on the geodesic segment γ from 1 to z and γ lies entirely in Ω,

$$G_r(x, z; \Omega) \leq C G_r(x, y; \Omega) G_r(y, z; \Omega).$$

The proof is virtually the same as in the unrestricted case.

Anderson-Schoen-Ancona Argument



Mark points $z_1, z_2, \ldots, z_{\varepsilon m}$ along geodesic segment $[y_0 x_0]$ such that the perpendicular geodesic through z_i divides $\mathbb{D}$ into two halfplanes L_i and R_i. Assume that z_i are spaced so that any geodesic segment from L_i to R_{i+1} passes within distance 2δ of z_i and z_{i+1}.

Note: $R_0 \supset R_1 \supset R_2 \supset \cdots$.

Anderson-Schoen-Ancona Argument

Define

$$u_0(z) = G_r(z,y)/G_r(x,y) \quad \text{and}$$
$$v_0(z) = G_r(z,y')/G_r(x,y')$$

Note:

- Ancona inequalities imply $u_0 \asymp v_0$ in L_0.
- Both u_0, v_0 are $r-$harmonic in R_0.
- Both u_0, v_0 are bounded in R_0.
- $u_0(x) = v_0(x) = 1$.

To Show: In R_n,

$$|u_0/v_0 - 1| = \left| \frac{u_n + \sum_{i=1}^n \varphi_i}{v_n + \sum_{i=1}^n \varphi_i} - 1 \right| \leq C'(1-\varepsilon)^n$$

Anderson-Schoen-Ancona Argument

Plan: Inductively construct $r-$harmonic functions φ_i, u_i, v_i in halfplane R_i such that

$$u_{i-1} = u_i + \varphi_i \quad \text{and} \quad u_{i-1} \geq \varphi_i \geq \varepsilon u_{i-1} \text{ in } A_i;$$
$$v_{i-1} = v_i + \varphi_i \quad \text{and} \quad v_{i-1} \geq \varphi_i \geq \varepsilon v_{i-1} \text{ in } A_i$$

This will imply

$$u_n \leq (1-\varepsilon)^n u_0$$
$$v_n \leq (1-\varepsilon)^n v_0$$
$$|u_n - v_n| \leq C(1-\varepsilon)^n (u+v)$$

$$\Longrightarrow |u_0/v_0 - 1| = \left| \frac{u_n + \sum_{i=1}^n \varphi_i}{v_n + \sum_{i=1}^n \varphi_i} - 1 \right| \leq C'(1-\varepsilon)^n$$

Anderson-Schoen-Ancona Argument

Assume that u_i, v_i, φ_i have been constructed. Since they are $r-$harmonic in R_i, Poisson Integral Formula implies

$$u_i(z) = \sum_{w \notin R_i} G_r(z, w; R_i) u_i(w),$$
$$v_i(z) = \sum_{w \notin R_i} G_r(z, w; R_i) v_i(w).$$

By construction, every geodesic segment from R_{i+1} to a point w not in R_i must pass within 2δ of z_{i+1}. Hence, Ancona inequalities imply

$$G_r(z, w; R_i) \asymp G_r(z, z_{i+1}; R_i) G_r(z_{i+1}, w; R_i) \quad \forall\, z \in R_{i+1}.$$

Anderson-Schoen-Ancona Argument

Consequently,

$$u_i(z) \asymp G_r(z, z_{i+1}; R_i) \sum_{w \notin R_i} G_r(z_{i+1}, w; R_i) u_i(w),$$

$$v_i(z) \asymp G_r(z, z_{i+1}; R_i) \sum_{w \notin R_i} G_r(z_{i+1}, w; R_i) u_i(w),$$

Thus, for small $\alpha > 0$

$$\varphi_{i+1}(z) = \alpha u_i(x) \frac{G_r(z, z_{i+1}; R_i)}{G_r(x, z_{i+1}; R_i)} = \alpha v_i(x) \frac{G_r(z, z_{i+1}; R_i)}{G_r(x, z_{i+1}; R_i)}$$

satisfies

$$\varepsilon u_i \leq \varphi_{i+1} \leq u_i$$

$$\varepsilon v_i \leq \varphi_{i+1} \leq v_i.$$

III. Local Limit Theorems: Hyperbolic Groups

Tomorrow:

Theorem: (Gouezel-Lalley) For any symmetric FRRW on a co-compact Fuchsian group,

$$P^1\{X_{2n} = 1\} \sim CR^{-2n}(2n)^{-3/2}.$$

Theorem: (Gouezel) This also holds for any nonelementary hyperbolic group. Moreover, for Fuchsian groups the hypothesis of symmetry is unnecessary.

Note: Same local limit theorem also holds for finitely generated Fuchsian groups Γ such that $\mathbb{H}/\Gamma$ has finite hyperbolic area and finitely many cusps.