# HARMONIC FUNCTIONS AND MARKOV CHAINS

## 1. Harmonic Functions

Let $(X_n)_{n\geq 0}$ be a Markov chain with (finite or countable) state space $\mathscr{X}$ and one-step transition probabilities $p(x,y)$. A real-valued function $h:\mathscr{X}\to\mathbb{R}$ is said to be *harmonic* at the site $x\in\mathscr{X}$ if

$$h(x)=\sum_{y\in\mathscr{X}} p(x,y)h(y)=E^x h(X_1). \tag{1}$$

If for some set $A\subset\mathscr{X}$ the function $h$ is harmonic at every $x\in A$, then $h$ is said to be *harmonic in $A$*; if $h$ is harmonic in $\mathscr{X}$ then we just say that $h$ is harmonic.

**Theorem 1.** *Assume that $h$ is harmonic in $A$, and let $T$ be a finite stopping time such that $X_n\in A$ for every $n<T$ (for instance, $T$ might be the first $n$ such that $X_n\notin A$). Then for every $n=0,1,2,\dots$ and every $x\in A$*

$$h(x)=E^x h(X_{T\wedge n}). \tag{2}$$

This should remind you of the Wald identities for random walks, and in fact the identity (2) can be used in much the same way as we used the Wald identities to solve first-passage problems for random walks. Some examples will be given in later sections of these notes.

*Proof.* Induction on $n$. The case $n=0$ is trivial, because in this case the identity (2) reduces to the obvious equality $h(x)=E^x h(X_0)$. For the induction step, assume that the identity holds for $n$; then by the law of total probability,

$$\begin{aligned}E^x h(X_{T\wedge(n+1)})&=E^x h(X_{T\wedge(n+1)})\mathbf{1}\{T\leq n\}+E^x h(X_{T\wedge(n+1)})\mathbf{1}\{T>n\}\\ &=E^x h(X_{T\wedge n})\mathbf{1}\{T\leq n\}+E^x h(X_{n+1})\mathbf{1}\{T>n\}.\end{aligned} \tag{3}$$

Now on the event $T>n$ it must be the case that $X_n\in A$, so by the Markov property,

$$\begin{aligned}E^x h(X_{n+1})\mathbf{1}\{T>n\}&=\sum_{y\in A} E^x h(X_{n+1})\mathbf{1}\{X_n=y\}\mathbf{1}\{T>n\}\\ &=\sum_{y\in A} E^x(h(X_{n+1})\,|\,X_n=y)P^x\{X_n=y \text{ and } T>n\}\\ &=\sum_{y\in A} E^y(h(X_1))P^x\{X_n=y \text{ and } T>n\}\\ &=\sum_{y\in A} h(y)P^x\{X_n=y \text{ and } T>n\}\\ &=E^x h(X_n)\mathbf{1}\{T>n\}.\end{aligned}$$

Combining this with (3) gives

$$\begin{aligned}E^x h(X_{T\wedge(n+1)}) &= E^x h(X_{T\wedge n})\mathbf{1}\{T\le n\} + E^x h(X_n)\mathbf{1}\{T>n\}\\ &= E^x h(X_{T\wedge n})\mathbf{1}\{T\le n\} + E^x h(X_{T\wedge n})\mathbf{1}\{T>n\}\\ &= E^x h(X_{T\wedge n}),\end{aligned}$$

and so the result follows by the induction hypothesis. □

**Definition 1.** A function $h(n,x)$ of both time $n=0,1,2,\dots$ and location $x\in\mathscr{X}$ is said to be *space-time harmonic* at $(n,x)$ if

$$h(n,x)=\sum_{y\in\mathscr{X}} h(n+1,y)p(x,y)=E^x h(n+1,X_1).$$

**Theorem 2.** *Assume that $h$ is space-time harmonic at every $(n,x)$ such that $x\in A$, and let $T$ be a stopping time such that $X_n\in A$ for every $n<T$. Then for every $n=0,1,2,\dots$,*

$$h(0,x)=E^x h(T\wedge n, X_{T\wedge n}). \tag{4}$$

*Proof.* The proof is nearly identical to that of Theorem 1. Details are left as an exercise. □

## 2. Harmonic Functions for Galton-Watson Processes

Let $Z_n$ be a Galton-Watson process with offspring distribution $(p_k)_{k=0,1,2,\dots}$ and initial condition $Z_0=1$. Thus, $Z_{n+1}=\sum_{i=1}^{Z_n}\xi_i^{(n+1)}$ where the random variables $\xi_i^{(n)}$ are independent, identically distributed with common distribution $P\{\xi_i^{(n)}=k\}=p_k$. Set

$$\varphi(t)=\sum_{k=0}^{\infty} t^k p_k \quad\text{and}\quad \mu=\sum_{k=0}^{\infty} kp_k.$$

**Proposition 3.** *If $0<\zeta\le 1$ is the smallest root of the fixed-point equation $\varphi(\zeta)=\zeta$ then the function $h(m)=\zeta^m$ is harmonic for the Galton-Watson process.*

*Proof.* Simple calculation. □

**Corollary 4.** *Assume that $p_0>0$ and that $\zeta<1$. Then $P\{extinction\}=\zeta$.*

We have already proved this by other methods earlier; here we will show that it can also be deduced from the identity (2). First, we must prove the following lemma concerning the behavior of the Galton-Watson process on the event that it does *not* reach extinction.

**Lemma 5.** *For any Galton-Watson process, with probability one either $Z_n=0$ eventually or $\lim_{n\to\infty} Z_n=\infty$.*

*Proof.* (Sketch) For any integer $k\ge 1$, at any time $n$ when $Z_n\le k$ there is conditional probability at least $p_0^k>0$ that $Z_{n+1}=0$. Consequently, there cannot be infinitely many times $n$ such that $1\le Z_n\le k$, because then there would be infinitely many chances to hit an event of probability $p_0^k$. Thus, for every $k=1,2,\dots$ there are only finitely many times $n$ when $Z_n=k$, and so either $Z_n=0$ eventually or $Z_n$ wanders off to $+\infty$. □

*Proof of Corollary 4.* Let $T_m$ be the first time $n$ that $Z_n \geq m$, or $T_m = \infty$ if there is no such $n$. By the preceding lemma, on the event $T_m = \infty$ it must be the case that $Z_n = 0$ eventually. By identity (2), for every $n = 1, 2, 3, \dots$

$$\zeta = E\zeta^{Z_{T_m \wedge n}}.$$

As $n \to \infty$ the random variable $\zeta^{Z_{T_m \wedge n}}$ converges to $\zeta^{T_m}$; since all of these random variables are uniformly bounded by 1 the bounded convergence theorem implies that

$$\zeta = E\zeta^{Z_{T_m}}.$$

As $m \to \infty$ the random variables $Z_{T_m}$ converge to $\infty$ on the event that the Galton-Watson process does *not* reach extinction, but converge to 0 on the event of extinction. Once again, by the bounded convergence theorem,

$$\zeta = \zeta^0 \times P(Z_n = 0 \text{ eventually}) + 0 \times P(Z_n \to \infty).$$

□

## 3. Birth-and-Death Chains

## 4. A Recurrence Criterion