STATISTICS 313: STOCHASTIC PROCESSES II
HOMEWORK ASSIGNMENT 8
DUE WEDNESDAY NOVEMBER 30

In the following problems , $W(t)$ is a standard Wiener process, $M(t)$ is the maximum up to time t, and for any $a > 0$, $\tau(a)$ is the first time that $W(t)$ visits a; thus,

$$M(t) = \max\{W(s) : s \le t\},$$
$$\tau(a) = \min\{t : W(t) = a\}.$$

Problem 1. Calculate each of the following probabilities. (Give your answers as rational numbers or decimals to at least 3 places. You might find a table of the standard normal cumulative distribution useful for this.)

(A) $P\{W(t) = 0 \text{ for some } 2 \le t \le 3\}$.
(B) $P\{W(2) > W(1) > W(3)\}$.

Problem 2. Show that for any $t > 0$ the random variable $M(t) - W(t)$ has the same distribution as the random variable $|W(t)|$. HINT: Begin by calculating

$$P\{M(t) \ge a \text{ and } W(t) \le a - b\}$$

for $a, b > 0$. You may find the reflection principle helpful.

Problem 3. Let $T_- = \max\{t < 1 : W_t = 0\}$ and $T_+ = \min\{t > 1 : W_t = 0\}$.

(A) Show that with probability one, $0 < T_- < 1 < T_+$.
(B) Prove that T_- is *not* a stopping time, but that T_+ is.
(C) Show that $P\{T_+ \le s\} = (2/\pi)\arccos(1/\sqrt{s})$ for all $s \ge 1$.
(D) Show that $P\{T_+ \ge t \text{ and } T_- \le s\} = (2/\pi)\arcsin\sqrt{s/t}$ for all $0 < s < 1 < t < \infty$.
(E) What is the density of T_-/T_+?

Problem 4. *Killed Brownian Motion.* Define T to be the first time that the Wiener process hits either $-1/2$ or $+1/2$. (Recall from class that $T < \infty$ almost surely.) For $-1/2 < x, y < 1/2$ and $t > 0$ define

$$p_t^*(x, y) = P^x\{W_t \in dy \text{ and } T > t\}. \tag{1}$$

(Here P^x is a probability measure under which W_t is a Brownian motion started at x.) This is a *sub*-probability density which is bounded above by the Gaussian density with mean 0 and variance t. Use an iterated version of the Reflection Principle to derive the following infinite series expansion for the density $p_t^*(x, y)$:

$$\begin{aligned} p_t^*(0, y) &= \sum_{k=-\infty}^{\infty} (-1)^k p_t((-1)^k y + k) \\ &= \sum_{k=-\infty}^{\infty} (-1)^k \exp\{-((-1)^k y + k)^2/2t\}/\sqrt{2\pi t}. \end{aligned} \tag{2}$$

Optional Problem. *Local Maxima of the Brownian Path.* A continuous function $f(t)$ is said to have a *local maximum* at $t = s$ if there exists $\varepsilon > 0$ such that

$$f(t) \leq f(s) \qquad \text{for all } t \in (s-\varepsilon, s+\varepsilon).$$

(A) Prove that for any fixed time $t 0$,

$$P\{W_t = M_t\} = 0.$$

HINT: Time reversal.

(B) For any two rational times $q_0 < q_1$ let $M(q_0, q_1) = \max_{t\in[q_0,q_1]} W_t$ be the maximum of the Brownian path in the time interval $[q_0, q_1]$. Prove that with probability one,

$$W_{q_0} < M(q_0, q_1) \quad \text{and} \quad W_{q_1} < M(q_0, q_1)$$

for *all* pairs of rationals q_0, q_1.

(C) Conclude that with probability one, the set of times at which the Brownian path $W(t)$ has a local maximum is dense in $[0, \infty)$.

(D) Prove that with probability one, for any two pairs of rational times $q_0 < q_1$ and $q_2 < q_3$,

$$M(q_0, q_1) \neq M(q_2, q_3).$$

HINT: Strong Markov property.

(E) Prove that, with probability one, the set of local maxima of the Brownian path $W(t)$ is *countable*.