STATISTICS 312: STOCHASTIC PROCESSES
HOMEWORK ASSIGNMENT 7
DUE WEDNESDAY NOVEMBER 23

Problem 1. *Birth-Death Chains.* A *birth-death* chain on the nonnegative integers $\mathbb{Z}_+$ is an irreducible Markov chain on $\mathbb{Z}_+$ for which only transitions to nearest neighbors are allowed. Thus, the nonzero transition probabilities are

$$\begin{aligned} p(x,x+1) &:= \beta_x > 0 && \text{for } x \geq 0 \\ p(x,x-1) &:= \alpha_x = 1-\beta_x > 0 && \text{for } x \geq 1 \\ p(0,0) &:= \alpha_0 = 1-\beta_0 > 0. \end{aligned}$$

(A) Check that the function f defined by

$$\begin{aligned} f(0) &= 0, \\ f(1) &= 1, \\ f(m) &= \sum_{k=0}^{m-1} \prod_{j=1}^{k} \frac{\alpha_j}{\beta_j} \end{aligned}$$

is harmonic in $D = \{1,2,3,\dots\}$. (Note: The value of the product when $k=0$ is 1.)

(B) Solve the gambler's ruin problem for the birth-death chain, that is, if $T = T_{0,K}$ is the first time that the Markov chain visits either 0 or K, then for each $0 \leq k \leq K$ find

$$P^k\{X_T = K\}.$$

(C) Use the result of (B) to show that the Markov chain is recurrent if and only if

$$\sum_{k=0}^{\infty} \prod_{j=1}^{k} \frac{\alpha_j}{\beta_j} = \infty.$$

Problem 2. A Queueing Model: Consider a 1-server queueing system in which jobs arrive at the occurrence times of a rate–λ Poisson process. The server can process two jobs simultaneously, and is active only when there are at least 2 jobs in the system (thus, if a job comes in when the queue is empty, it must wait until another job arrives before its service begins). When there are at least 2 jobs in the system, the server works on the two that arrived first; the time needed to finish these 2 jobs is exponential with parameter β. Let X_t =number of jobs in the system at time t.

(A) What is the Q–matrix? NOTE: Before you go on to part (B), you might want to check with Si Tang to be sure you have the right answer.

(B) Show that if $\beta > 2\lambda$ then there is a stationary distribution, and find it.

Problem 3. Reversibility: Let $\mathbb{P}_t = (p_t(x,y))_{x,y\in\mathscr{X}}$ be the transition semigroup of a continuous-time Markov chain $\{X_t\}_{t\geq 0}$ on a finite state space $\mathscr{X}$, and let $\mathbb{Q} = (q(x,y))_{x,y\in\mathscr{X}}$ be the associated Q—matrix (see notes). Thus,

$$\mathbb{P}_t = \exp\{t\mathbb{Q}\}$$

where exp is the matrix exponential function. Say that the Markov chain is *reversible* if there exists a probability distribution $\{w_x\}_{x\in\mathscr{X}}$ on $\mathscr{X}$ such that for any two states x, y,

$$w_x q(x,y) = w_y q(y,x). \tag{1}$$

These equations are called the *detailed balance equations*.

(A) Prove that if the detailed balance equations hold, then for every $t \geq 0$ and every pair of states x, y,

$$w_x p_t(x,y) = w_y p_t(y,x).$$

HINT: For each integer $m \geq 1$ denote by $a^m(x,y)$ the x, y entry of the matrix $\mathbb{Q}^m$. Show (by induction on m) that for every $m \geq 1$,

$$w_x a^m(x,y) = w_y a^m(y,x).$$

Then use the matrix equation

$$\mathbb{P}_t = \exp\{t\mathbb{Q}\} = \sum_{m=0}^{\infty} \frac{t^m \mathbb{Q}^m}{m!}.$$

(B) Prove that if the detailed balance equations hold, then the probability distribution $\{w_x\}_{x\in\mathscr{X}}$ on $\mathscr{X}$ is a stationary distribution.

(C) A continuous-time Markov chain on a finite interval $\mathscr{X} = \{0,1,2,\ldots,K\}$ is said to be a *birth-and-death* chain if its Q—matrix is such that

$$\begin{aligned} q(x,x+1) &> 0 \quad \text{for all } 0 \leq x < K; \\ q(x,x-1) &> 0 \quad \text{for all } 0 < x \leq K; \text{ and} \\ q(x,y) &= 0 \quad \text{otherwise.} \end{aligned}$$

Prove that every birth-and-death chain on a finite interval $\mathscr{X} = \{0,1,2,\ldots,K\}$ is reversible, and write a formula for the stationary distribution w.

Problem 4. Yule Process: The *Yule process* with rate parameter $\beta > 0$ is the continuous-time Markov chain on the set $\mathscr{X} = \mathbb{N}$ of positive integers with Q—matrix

$$\begin{aligned} q(x,x+1) &= \beta x \quad \text{for all } x \geq 1, \\ q(x,y) &= 0 \quad \text{otherwise.} \end{aligned}$$

(A) Verify that the transition probabilities are given by the following formula:

$$p_t(j,k) = \binom{k-1}{j-1} e^{-j\beta t}(1 - e^{-\beta t})^k.$$

HINT: Check that this satisfies the Kolmogorov backward equations.

(B) The state X_t of a Yule process can be viewed as the size of a branching population in which individuals wait exponential-1 times and then fission, producing 1 new particle at each fission event. For any $t \geq s > 0$, let $N(t;s)$ be the number of individuals in the population that were born after time $t-s$. Find the distribution of $N(t;s)$, assuming that $X_0 = 1$ (i.e., that the population starts with just one individual).

(C) Let X_t and Y_t be *independent* Yule processes, both with rate parameter $\beta > 0$, and with initial states $X_0 = x$ and $Y_0 = y$. Show that $Z_t = X_t + Y_t$ is a Yule process with rate parameter $\beta > 0$ and initial state $Z_0 = x + y$.

(D) Use the results of (A) and (C) to conclude that the conditional distribution of X_t given that $Z_t = N$ is a hypergeometric distribution. (You figure out the parameters!)