STATISTICS 312: STOCHASTIC PROCESSES
HOMEWORK ASSIGNMENT 4
DUE MONDAY OCTOBER 24

Problem 1. Residual Lifetime Distribution. Let $\{S_n\}_{n\geq 0}$ be a discrete renewal process with interoccurrence time distribution $\{p_k\}_{k\geq 1}$. Thus,

$$S_n = \sum_{j=1}^{n} X_j$$

where $X_1, X_2, \ldots$ are independent, identically distributed with distribution $P\{X_1 = k\} = p_k$. Assume that the distribution $(p_{k\,k\geq 1})$ satisfies the hypotheses of the Erdös-Feller-Pollard theorem. For each $m = 0, 1, 2, \ldots$ let $N(m) =$ smallest n such that $S_n > m$, and set

$$\begin{aligned} R_m &= S_{N(m)} - m \quad \text{and} \\ A_m &= m - S_{N(m)-1}. \end{aligned}$$

These random variables are called the *residual lifetime* and the *age* at time m. This problem concerns their distributions.

(A) Use the Erdös-Feller-Pollard theorem to show that if $EX_1 = \sum_{k=1}^{\infty} kp_k = \mu < \infty$ then the residual lifetime random variable R_m has a distribution that converges as $m \to \infty$, that is, show that for each $k \geq 1$ the limit

$$\lim_{m\to\infty} P\{R_m = k\} := g_k$$

exists, and identify the limit distribution $(g_k)_{k\geq 1}$.

(B) Show that if $EX_1 = \sum_{k=1}^{\infty} kp_k = \mu < \infty$ then for each $k \geq 1$ and each $\ell \geq 0$,

$$\lim_{m\to\infty} P\{R_m = k \text{ and } A_m = \ell\} := h(k,\ell)$$

exists, and identify $h(k,\ell)$.

(C) Fix an integer $k \geq 1$, and set $v(m) = P\{R_m = k\}$. Show that the sequence $v(m)$ satisfies a *renewal equation*

$$v(m) = b(m) + \sum_{n=1}^{m} p_n v(m-n)$$

and identify the sequence $b(m)$. HINT: Condition on the first step of the random walk.

(D) Let $V(z)$, $B(z)$, and $F(z)$ be the generating functions of the sequences $(v_m)_{m\geq 0}$, $(b_k)_{k\geq 1}$, and $(p_n)_{n\geq 1}$, respectively. Find an algebraic equation that exhibits $V(z)$ as a rational expression in the other two functions $B(z)$ and $F(z)$.

(E) Consider the special case where $p_1 = p_2 = \frac{1}{2}$. Use the result of part (C) to find an exact formula for $P(R_m = k)$.

Problem 2. The Fibonacci Sequence: The Fibonacci sequence – probably discovered in India over 1000 years ago, but introduced to Western mathematics by Leonardo of Pisa, known as *Fibonacci* – is defined by the inductive rule

$$a_1 = a_2 = 1; \tag{1}$$
$$a_{n+2} = a_{n+1} + a_n.$$

Thus, the sequence begins $1,1,2,3,5,8,13,21,\cdots$. The Fibonacci numbers occur in unexpected places, for instance (see Wikipedia entry for *Conifer cone*)

> The members of the pine family (pines, spruces, firs, cedars, larches, etc.) have cones that are imbricate with scales overlapping each other like fish scales. These are the "archetypal" cone. *The scales are spirally arranged in Fibonacci number ratios.*

Define

$$A(z) := \sum_{n=1}^{\infty} a_n z^n.$$

Use the convolution equation (1) to obtain a quadratic functional equation for $A(z)$. Solve this equation, and then use the method of partial fractions to obtain an exact formula for the nth Fibonacci number. Use this to show that for a suitable constant C,

$$a_n \sim C\theta^n$$

where θ is the *golden ratio.*

Problem 3. A Queueing Model. At each time $n = 1,2,3,\ldots$ a random number V_n of passengers arrive at a bus stop. Assume that the random variables $V_1, V_2, \ldots$ are independent, identically distributed Bernoulli(p) for some value of $p \in (0,1)$. Buses arrive at the occurrence times of an arithmetic renewal process with interoccurrence time distribution $F = \{f_k\}_{k\geq 1}$. Whenever a bus arrives, all passengers at the stop board the bus and are then carted away. (Thus, each bus has infinite capacity.) Assume that the inter-arrival time distribution $F = \{f_k\}_{k\geq 1}$ satisfies the hypotheses of the Erdös-Feller-Pollard theorem.

(A) Let $Y_n = Y(n)$ be the number of passengers waiting at the bus stop at time n. (Note: Use the convention that at any time n that a bus arrives, $Y_n = 0$.) Fix $k \geq 0$. Use the Erdös-Feller-Pollard theorem to show that

$$\lim_{n\to\infty} P\{Y(n) = k\} := g(k)$$

exists. Identify the limit distribution $g(k)$ in terms of the distribution F and the arrival rate λ.

(B) What is the mean number of passengers waiting in steady state?

(C) What is the mean time, in steady state, that a passenger spends waiting for a bus to arrive?