STATISTICS 312: STOCHASTIC PROCESSES
HOMEWORK ASSIGNMENT 3
DUE MONDAY OCTOBER 17

Problem 1. Let be a $p:q$ random walk on the integers $\mathbb{Z}$ started at $S_0 = 0$, i.e., let

$$S_n = \sum_{i=1}^{n} X_i$$

where the steps X_i are independent, identically distributed with common distribution

$$P\{X_i = +1\} = p \quad \text{and} \quad P\{X_i = -1\} = q = 1 - p.$$

For each nonnegative integer m, define $\tau(m)$ to be the smallest $n \in \mathbb{N}$ such that $S_n = m$ (or $\tau(m) = \infty$ if there is no such n), and for any integers $A, B \geq 1$ define $T = T_{-A,B}$ to be the smallest $n \in \mathbb{N}$ such that $S_n = -A$ or $S_n = B$.

(A) Use the First Wald Identity and the Monotone Convergence Theorem to show that if $p > \frac{1}{2}$ then $E\tau(1) = 1/(p-q) = 1/(2p-1)$.

(B) Use the Third Wald Identity with $e^{\theta} = q/p$ to calculate

$$P\{S_T = B\}.$$

Problem 2. Let $X_1, X_2, \dots$ be independent, identically distributed random variables with the geometric distribution

$$P\{X_i = k\} = (1-r)r^k \quad \text{for } k = 0, 1, 2, \dots$$

for some fixed $0 < r < 1$, and let $S_n = \sum_{i=1}^{n} X_i$ be the random walk with steps X_i. For each integer $m = 1, 2, 3, \dots$ let T_m be the smallest integer n such that $S_n \geq m$.

(A) Show that for each $m = 1, 2, \dots$ the *overshoot* random variable $S_{T_m} - m$ has the geometric distribution

$$P\{S_{T_m} - m = k\} = (1-r)r^k \quad \text{for } k = 0, 1, 2, \dots.$$

(B) Calculate $E\,T_m$.

(C) For each $0 < r < 1$ and each $m = 1, 2, \dots$, find the probability that at some time $n \geq 0$,

$$S_n - n = -m.$$

NOTE: The form of the solution should change at $r = 1/2$.

Problem 3. Let $S_n = \sum_{i=1}^{n} \xi_i$ be the random walk on the integers that at each step makes jumps of size +1 or +2, with probabilities p and q, respectively. (That is, the increments ξ_i are independent, identically distributed random variables such that $P\{\xi_i = +1\} = p$ and $P\{\xi_i = +2\} = q = 1-p$.) For each integer $m \geq 0$, let $u(m)$ be the probability that $S_n = m$ for some $n \geq 1$. Find an explicit formula for $u(m)$, in two ways:

(A) Generating functions: Define $U(z) = \sum_{m=0}^{\infty} u(m) z^m$. Show that

$$U(z) = \frac{1}{1-F(z)} \quad \text{where} \quad F(z) = pz + qz^2,$$

and then use the method of partial fraction decomposition to express $U(z)$ as the sum of two geometric series.

(B) Wald III: Define $\tau(m)$ =smallest n such that $S_n \geq m$. Use Wald III with the right choice of e^{θ}. NOTE: It is believed that the axioms of mathematics are consistent, so in principle your answers to (A) and (B) should be the same.